

ASYMPTOTIC NONPARAMETRIC CONFIDENCE INTERVALS
FOR THE RATIO OF SCALE PARAMETERS IN BALANCED
ONE-WAY RANDOM EFFECTS MODELS

BY

DAVID JOHN GROGEL

A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1983

ACKNOWLEDGMENTS

I would like to express my deepest appreciation to all those who have assisted me during the preparation of this dissertation and throughout my time in graduate school. In particular I wish to thank Dr. Dennis Wackerly, chairman of my research committee, for his help with this research and for his guidance and kindness as my advisor. I would also like to thank Dr. P. V. Rao, for his frequent contributions to this research, and Dr. Richard Scheaffer and the entire Department of Statistics, for the encouragement, support, and friendships they have provided.

Further, I would like to thank my parents, Mr. and Mrs. Richard Groggel, for their ever-present love and support, and my in-laws, Mr. and Mrs. Warren Rubin, for their love and support over the past five years. Finally, special thanks goes to my wife Kathy for her help with the typing of this dissertation but especially for the love, patience, and encouragement she constantly offers.

TABLE OF CONTENTS

	PAGE
ACKNOWLEDGMENTS.....	ii
ABSTRACT.....	iv
CHAPTER	
ONE INTRODUCTION.....	1
TWO CONFIDENCE INTERVALS USING U-STATISTICS.....	7
2.1 General Theory of U-Statistics.....	7
2.2 Confidence Intervals for the Intraclass Correlation Coefficient.....	9
THREE CONFIDENCE INTERVALS USING MODIFIED ANSARI-BRADLEY STATISTICS.....	17
3.1 Model and Formation of Pseudo-Samples.....	17
3.2 Asymptotic Distribution of the Ansari-Bradley Statistic Using Pseudo-Samples.....	22
3.3 Asymptotic Confidence Intervals Using the Modified Ansari-Bradley Statistic.....	39
FOUR MONTE CARLO STUDY.....	44
FIVE SUMMARY.....	79
APPENDICES	
A VARIANCES AND COVARIANCE OF U_1 AND U_2	82
B THE RELATIONSHIP BETWEEN U_1 , U_2 , MST, AND MSE.....	85
C A CONSISTENT ESTIMATE FOR σ_T	89
D DERIVATION OF ENDPOINTS IN CHI-SQUARE PROCEDURE.....	93
E C AND C* TERMS.....	97
REFERENCES.....	125
BIOGRAPHICAL SKETCH.....	127

Abstract of Dissertation Presented to the Graduate School
of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

ASYMPTOTIC NONPARAMETRIC CONFIDENCE INTERVALS
FOR THE RATIO OF SCALE PARAMETERS IN BALANCED
ONE-WAY RANDOM EFFECTS MODELS

By

David John Groggel

August, 1983

Chairman: Dennis D. Wackerly
Major Department: Statistics

This dissertation examines the problem of estimating the proportion of variability in the responses from a balanced one-way random effects model that can be attributed to the treatment effect. Methods are derived that do not require the classical assumptions of normality of both the treatment and error effects.

Asymptotic confidence intervals are derived from functions of U-statistics that possess either an asymptotic normal distribution or asymptotic chi-square distribution. These methods require the effects to have continuous distributions with zero means and finite fourth moments.

Asymptotic confidence intervals are also derived based on the asymptotic normality of modified versions of the Ansari-Bradley two-sample scale statistic. Pseudo-samples of observations that are asymptotically equivalent to samples of the effects are formed using either sample means or sample medians. The Ansari-Bradley statistic

calculated using these samples is shown to have an asymptotic normal distribution and intervals are formed following a procedure of Sen ([1966] Annals of Mathematical Statistics 37, 1759-1770). In forming these intervals a representation of the Ansari-Bradley statistic developed by Bhattacharyya ([1977] Journal of the American Statistical Association 72, 459-463) is used. The construction of these intervals requires the effects to have continuous distributions that are symmetric about zero and that differ only by a scale parameter. Other assumptions on the distributions of the effects are needed depending on whether sample means or sample medians are used to form the pseudo-samples.

A Monte Carlo study was performed to compare intervals formed by these methods with the classical normal theory intervals and intervals based on jackknifed U-statistics as derived by Arvesen ([1969] Annals of Mathematical Statistics 40, 2076-2100). The study shows that the intervals based on functions of U-statistics are poor while the intervals based on the modified Ansari-Bradley statistics are nearly always comparable to and, in some cases, superior to the normal theory and Arvesen intervals.

CHAPTER ONE INTRODUCTION

The balanced one-way random effects model,

$$z_{ij} = \mu + \alpha_i + \epsilon_{ij} \quad i = 1, 2, \dots, k, \quad j = 1, 2, \dots, n,$$

has been studied and analyzed by many people. In this model, μ is an unknown constant and the ϵ_{ij} and α_i are independent samples of independent observations from continuous populations. The majority of the research concerning this model has been done under the classical assumptions that the ϵ_{ij} (commonly called the error effects) and α_i (commonly called the treatment effects) have normal distributions with zero means and variances σ_ϵ^2 and σ_α^2 respectively. In this dissertation, the model is studied under more general assumptions concerning the distributions of these effects.

In the classical case, the test of hypothesis that is usually of interest is a test concerning the magnitude of σ_α^2 . The test is usually of the form

$$(1.1) \quad H_0: \sigma_\alpha^2 = 0 \quad H_a: \sigma_\alpha^2 > 0$$

or

$$(1.2) \quad H_0: \sigma_\alpha^2 \leq c\sigma_\epsilon^2 \quad H_a: \sigma_\alpha^2 > c\sigma_\epsilon^2,$$

where c is some specified constant.

In some instances, particularly applications in genetics and the social sciences, an estimate of $\sigma^2_\alpha / (\sigma^2_\alpha + \sigma^2_\epsilon)$ is desired. This quantity is commonly known as the intraclass correlation coefficient and is denoted by ρ . In most cases, an estimate of ρ is more informative than testing the hypotheses in either (1.1) or (1.2). An estimate provides information about the actual relative magnitudes of the variance components rather than just a conclusion that H_0 should or should not be rejected.

As an example of where an estimate of ρ is useful, consider the problem described in Snedecor and Cochran (1967, Example 10.13.1). In a study involving Poland China swine, two boars were taken from each of four litters. All of the litters had the same sire and all eight boars were fed a standard ration from weaning to a weight of about 225 pounds. The response of interest was the average daily weight gain. The component σ^2_α represents the variability in weight gain that was due to the genetic differences in the litters while σ^2_ϵ represents the variability in weight gain due to non-genetic factors. The ratio $\sigma^2_\alpha / (\sigma^2_\alpha + \sigma^2_\epsilon)$ is the proportion of the total variability that can be attributed to the genetic differences in the litters.

Scheffe' (1959), as well as many others, describes the procedures for testing the hypotheses in (1.1) and (1.2) as well as the form of an exact confidence interval for ρ . Both the test procedure and confidence interval construction use the mean squares from the usual analysis of variance table and percentiles of the F-distribution.

Scheffe' shows that these procedures are not robust if the assumptions of the normality of the effects are violated. It is therefore desirable to have procedures that can be used to perform tests

and construct confidence intervals for the parameters associated with the balanced one-way random effects model which can be used when the assumptions of the normality of the effects is in doubt.

The analysis of the random effects model without the normality assumptions has not been researched nearly as much as the classical case. Govindarajulu and Deshpande' (1972) studied the case in which the ϵ_{ij} are independent and identically distributed with continuous distribution function $F(x)$ and the α_i are independent with distribution functions $G_i(x)$. In this case, it is not necessary that the expectations of the α_i all be equal. Assuming, without loss of generality, that $\mu = 0$ the authors examined the hypotheses

$$(1.3) \quad \begin{aligned} H_0: G_i(x) &= \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} && \text{for every } i \\ H_a: G_i(x) &\text{ is nontrivial for at least one } i \end{aligned}$$

and derived the locally most powerful rank test by considering the alternative hypothesis

$$H_\Delta: Z_{ij} = \Delta\alpha_i + \epsilon_{ij} \quad \text{for small positive } \Delta.$$

The hypotheses in (1.3) are analogous to the hypotheses in (1.1) since in both cases the null hypothesis states that the α_i do not contribute to the response variable Z_{ij} and the alternative hypothesis states that at least one α_i contributes to the response. Govindarajulu (1975) looked at the same hypotheses under the more restrictive assumption that $G_i(x) = G(x)$ for every i . Both of these papers considered the unbalanced one-way random effects model, that is, $j = 1, 2, \dots, n_i$ for $i = 1, 2, \dots, k$.

For the balanced one-way model, Arvesen (1969) and Arvesen and Schmitz (1970) used jackknifing techniques on appropriate U-statistics to develop procedures for testing hypotheses and forming confidence intervals for functions of σ_ϵ^2 and σ_α^2 . This work was later extended to the unbalanced model by Arvesen and Layard (1975). The procedures require the distributions of the ϵ_{ij} and α_i to be continuous with zero means. The procedures also assume finite fourth moments in the balanced model and finite moments of at least order six in the unbalanced model.

Shoemaker (1981) examined some estimation and testing problems using the concept of mid-variances in the balanced model where the effects are assumed to have continuous, symmetric distributions.

In Chapter Two of this dissertation, two methods of constructing asymptotic confidence intervals for ρ based on the theory of U-statistics are described. Section 2.1 gives a brief review of some of the basic results concerning U-statistics. In Section 2.2 a method using U-statistics similar to those used by Arvesen (1969) is described. This method was developed before the work of Arvesen was known to exist. However, the confidence coefficient used for the intervals in Section 2.2 is derived in a way different than that presented by Arvesen. As in Arvesen's work, the method in Section 2.2 requires the ϵ_{ij} and α_i to be independent random samples of independent observations from continuous distributions with zero means and finite fourth moments. Also in Section 2.2, an asymptotic confidence interval for ρ is derived using a quadratic form (involving two U-statistics) to construct a statistic with an asymptotic chi-square distribution with two degrees of freedom.

In Chapter Three we work with scale parameters rather than variances. The distribution of a random variable X is said to have a scale parameter δ ($0 < \delta < \infty$) if X has a distribution function of the form $F(x/\delta)$ where $F(x)$ is the distribution function of a random variable Y and the form of $F(x)$ does not depend on δ . In other words, X/δ has the same distribution as Y . The advantage of working with scale parameters is that they may exist for random variables for which variances (and thus standard deviations) do not exist. For those random variables where both a scale parameter and a standard deviation exist, a scale parameter is always a constant multiple of the standard deviation.

In Chapter Three the ϵ_{ij} and α_i are assumed to be independent samples of independent observations from continuous distributions with distribution functions $F(x) = D(x/\delta_1)$ and $G(x) = D(x/\delta_2)$ respectively. That is, δ_1 is a scale parameter for the ϵ_{ij} and δ_2 is a scale parameter for the α_i . It is also assumed that both distributions are symmetric about zero with densities that are bounded and have a bounded first derivative. In Section 3.1 the Ansari-Bradley two-sample scale statistic (Ansari and Bradley 1960) is described. Modified versions of the Ansari-Bradley statistic, one involving the use of sample means and another involving sample medians, are shown to have asymptotic normal distributions in Section 3.2. In Section 3.3 these statistics are used to form asymptotic confidence intervals for $\delta_2^2/(\delta_1^2 + \delta_2^2)$. In those situations where both scale parameters and variances of the effects exist, this quantity is numerically equivalent to ρ .

In Chapter Four we present a summary of a Monte Carlo study that compares the lengths and observed confidence coefficients of intervals constructed using normal theory as in Scheffe' (1959), Arvesen's (1969)

U-statistics, U-statistics as described in Chapter Two, and the modified Ansari-Bradley statistics as described in Chapter Three. Chapter Five contains a summary.

Throughout this dissertation we use the symbol \equiv to denote equal by definition. Also, unless otherwise specified, sums involving i are from 1 to k, sums involving j are from 1 to n, and integrals are over the region $(-\infty, \infty)$.

CHAPTER TWO
CONFIDENCE INTERVALS USING U-STATISTICS

2.1 General Theory of U-Statistics

The theory of U-statistics was first developed by Hoeffding (1948). For the convenience of the reader, in this section we state without proof some results and theorems due to Hoeffding which we will utilize in the discussions which follow.

Let $\underline{Z}_1, \underline{Z}_2, \dots, \underline{Z}_m$ be independent, identically distributed random vectors and let $h(\underline{Z}_1, \underline{Z}_2, \dots, \underline{Z}_s)$ be a function of $s (\leq m)$ of these vectors. A U-statistic has the form

$$U_m = U(\underline{Z}_1, \underline{Z}_2, \dots, \underline{Z}_m) = \binom{m}{s}^{-1} \sum_{V \in V} h(\underline{Z}_{v_1}, \underline{Z}_{v_2}, \dots, \underline{Z}_{v_s}),$$

where V is the set of all distinct subsets of integers, (v_1, v_2, \dots, v_s) , taken without replacement and without regard to order from $(1, 2, \dots, m)$.

It is easily seen that U_m is an unbiased estimate of the parameter $\Delta = E[h(\underline{Z}_1, \underline{Z}_2, \dots, \underline{Z}_s)]$. The function h is assumed to be symmetric in its arguments (it can be made so if it is not) and is known as the kernel. The value of s is the smallest possible sample size for which an unbiased estimate of Δ exists and is referred to as the degree of the kernel.

Define

$$h^c(\underline{z}_1, \underline{z}_2, \dots, \underline{z}_c) = E[h(\underline{Z}_1, \underline{Z}_2, \dots, \underline{Z}_s) | \underline{Z}_1 = \underline{z}_1, \underline{Z}_2 = \underline{z}_2, \dots, \underline{Z}_c = \underline{z}_c]$$

and

$$(2.1.1) \quad \xi_c = E([h^c(z_1, z_2, \dots, z_c)]^2) - \Delta^2$$

for $c = 1, 2, \dots, s$. The quantity ξ_c can also be written as

$$(2.1.2) \quad \xi_c = Cov[h(z_{v_1}, z_{v_2}, \dots, z_{v_s})h(z_{v'_1}, z_{v'_2}, \dots, z_{v'_s})],$$

where $v = (v_1, v_2, \dots, v_s)'$ and $v' = (v'_1, v'_2, \dots, v'_s)'$ are subsets of the integers $(1, 2, \dots, m)$ with exactly c integers in common.

The variance and asymptotic distribution of a U-statistic are given in the following results and theorem.

Result 2.1.1 (Hoeffding 1948, Equation 5.13). If $E[h^2(z_1, z_2, \dots, z_s)] < \infty$, then the variance of U_m is

$$\text{Var}(U_m) = \binom{m}{s}^{-1} \sum_{c=1}^s \binom{s}{c} \binom{m-s}{s-c} \xi_c.$$

Result 2.1.2 (Hoeffding 1948, Equation 5.23). If $E[h^2(z_1, z_2, \dots, z_s)] < \infty$, then $\lim_{m \rightarrow \infty} m\text{Var}(U_m) = s^2 \xi_1$.

Theorem 2.1.1 (Hoeffding 1948, Theorem 7.1). If $E[h^2(z_1, z_2, \dots, z_s)] < \infty$ and $\xi_1 > 0$, then $m^{1/2}(U_m - \Delta) \xrightarrow[m \rightarrow \infty]{d} N(0, s^2 \xi_1)$.

For $w = 1, 2, \dots, g$, let $U_m^{(w)}$ be U-statistics all defined on the same m vectors with degrees $s(w)$, kernels h_w , and expectations $\Delta(w)$. For any two of these U-statistics, say $U_m^{(1)}$ and $U_m^{(2)}$, define

$$(2.1.3) \quad \xi_c^{(1,2)} = E[h_1(z_{v_{11}}, z_{v_{12}}, \dots, z_{v_{1s(1)}})h_2(z_{v_{21}}, z_{v_{22}}, \dots, z_{v_{2s(2)}})] - \Delta(1)\Delta(2),$$

where $v_1 = (v_{11}, v_{12}, \dots, v_{1s(1)})'$ and $v_2 = (v_{21}, v_{22}, \dots, v_{2s(2)})'$ are

subsets of the integers $(1, 2, \dots, m)$ with exactly c integers in common. The covariance of these U-statistics and their joint asymptotic distribution are described in the following results and theorem.

Result 2.1.3 (Hoeffding 1948, Equation 6.5). If

$E(h_1^2) < \infty$, $E(h_2^2) < \infty$, and $s(2) < s(1)$, then the covariance

of $U_m^{(1)}$ and $U_m^{(2)}$ is such that

$$\text{Cov}[U_m^{(1)}, U_m^{(2)}] = \binom{m}{s(2)}^{-1} \sum_{c=1}^{s(2)} \binom{s(1)}{c} \binom{m-s(1)}{s(2)-c} \xi_c^{(1,2)}.$$

Result 2.1.4 (Hoeffding 1948, Page 304). Under the same conditions as in Result 2.1.3, $\lim_{m \rightarrow \infty} m \text{Cov}[U_m^{(1)}, U_m^{(2)}] = s(1)s(2)\xi_1^{(1,2)}$.

Theorem 2.1.2 (Hoeffding 1948, Theorem 7.1). If $E(h_w^2) < \infty$ for $w = 1, 2, \dots, g$, then

$$m^{1/2}([U_m^{(1)-\Delta(1)}], [U_m^{(2)-\Delta(2)}], \dots, [U_m^{(g)-\Delta(g)}]) \xrightarrow[m \rightarrow \infty]{d} N_g(0, \Delta),$$

where Δ is a g by g matrix with elements $\Delta_{ij} = s(i)s(j)\xi_1^{(i,j)}$.

In a later paper Hoeffding proved the following theorem concerning the asymptotic convergence of a U-statistic.

Theorem 2.1.3 (Hoeffding 1961). If $E[|h(z_1, z_2, \dots, z_s)|] < \infty$, then $U_m \xrightarrow[m \rightarrow \infty]{a.s.} \Delta$.

2.2 Confidence Intervals for the Intraclass Correlation Coefficient

Consider the balanced one-way random effects model

$$z_{ij} = \mu + \alpha_i + \epsilon_{ij} \quad i = 1, 2, \dots, k, \quad j = 1, 2, \dots, n,$$

where the ϵ_{ij} are independent random variables with a continuous

distribution with mean zero and finite fourth moment and the α_i are independent random variables with a continuous distribution (not necessarily the same family as the distribution of the ϵ_{ij}) which also has mean zero and finite fourth moment. The ϵ_{ij} and α_i are assumed to be independent of each other and the variances of the two distributions are denoted by σ_ϵ^2 and σ_α^2 respectively. The parameter μ is an unknown constant.

In the work that follows in this section, the number of observations per treatment, n , remains fixed as the number of treatments, k , increases to infinity. Due to the structure of the model this is sufficient to obtain, at least theoretically, unlimited knowledge about both the ϵ_{ij} and α_i .

Let $Z_i = (z_{i1}, z_{i2}, \dots, z_{in})'$, for $i = 1, 2, \dots, k$, be k independent and identically distributed vectors. On these k vectors we define two U-statistics,

$$(2.2.1) \quad U_1 = k^{-1} \sum_i h_1(Z_i), \\ \text{where } h_1(Z_i) = \binom{n}{2}^{-1} \sum_{j < j'} (z_{ij} - z_{ij'})^2$$

and

$$(2.2.2) \quad U_2 = \binom{k}{2}^{-1} \sum_{i < i'} h_2(Z_i, Z_{i'}), \\ \text{where } h_2(Z_i, Z_{i'}) = n^{-2} \sum_{jj'} (z_{ij} - z_{i'j'})^2.$$

These U-statistics are unbiased estimates for the expectations of their respective kernels which are

$$(2.2.3) \quad \begin{aligned} E(U_1) &= E[h_1(Z_1)] = E[(Z_{11}-Z_{12})^2] \\ &= E(\epsilon_{11}-\epsilon_{12})^2 = 2\sigma_\epsilon^2 \end{aligned}$$

and

$$(2.2.4) \quad \begin{aligned} E(U_2) &= E[h_2(Z_1, Z_2)] = E[(Z_{11}-Z_{21})^2] \\ &= E(\alpha_1-\alpha_2+\epsilon_{11}-\epsilon_{21})^2 = 2\sigma_\alpha^2 + 2\sigma_\epsilon^2. \end{aligned}$$

If $\phi_4 \equiv E(\epsilon_{11}^4)$ and $\eta_4 \equiv E(\alpha_1^4)$ are finite, and thus $E(h_1^2) < \infty$ and $E(h_2^2) < \infty$, Results 2.1.2 and 2.1.4 imply (see Appendix A) that

$$(2.2.5) \quad \lim_{k \rightarrow \infty} kVar(U_1) = 4n^{-1}[\phi_4 + \sigma_\epsilon^4(3-n)(n-1)^{-1}] \equiv \sigma_{11},$$

$$(2.2.6) \quad \lim_{k \rightarrow \infty} kVar(U_2) = 4(\eta_4 + \phi_4/n - \sigma_\alpha^4/n - \sigma_\epsilon^4/n + 4\sigma_\alpha^2\sigma_\epsilon^2/n) \equiv \sigma_{22},$$

and

$$(2.2.7) \quad \lim_{k \rightarrow \infty} kCov(U_1, U_2) = 4n^{-1}(\phi_4 - \sigma_\epsilon^4) \equiv \sigma_{12}.$$

Since $E(\epsilon_{11}^4) = \phi_4 > [E(\epsilon_{11}^2)]^2 = \sigma_\epsilon^4$, it is clear that, for large k , U_1 and U_2 are positively correlated.

Using Theorem 2.1.2 we can describe the asymptotic distributions of U_1 and U_2 in the following theorem.

Theorem 2.2.1. If U_1 and U_2 are U-statistics as defined in (2.2.1) and (2.2.2) and if σ_{11} , σ_{22} , and σ_{12} are as defined in (2.2.5) through (2.2.7), then

$$(k^{1/2}[U_1 - 2\sigma_\epsilon^2], k^{1/2}[U_2 - (2\sigma_\alpha^2 + 2\sigma_\epsilon^2)]) \xrightarrow[k \rightarrow \infty]{d} N_2(\underline{0}, \underline{\Lambda}),$$

where

$$(2.2.8) \quad \underline{\Lambda} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}.$$

Using Theorem 2.1.3, (2.2.3), and (2.2.4), we know $U_1 \xrightarrow{k \rightarrow \infty} 2\sigma_\epsilon^2$ and

$U_2 \xrightarrow{k \rightarrow \infty} 2\sigma_\alpha^2 + 2\sigma_\epsilon^2$. It then follows that $U_1/U_2 \xrightarrow{k \rightarrow \infty} \sigma_\epsilon^2/(\sigma_\alpha^2 + \sigma_\epsilon^2)$

which is equal to $1 - \sigma_\alpha^2/(\sigma_\alpha^2 + \sigma_\epsilon^2)$. Thus, $1 - U_1/U_2$ is a strongly consistent point estimate for the intraclass correlation coefficient, $\rho = \sigma_\alpha^2/(\sigma_\alpha^2 + \sigma_\epsilon^2)$.

It is useful to note at this point that U_1 and U_2 are related to quantities encountered in the classical normal theory one-way analysis of variance. If MST and MSE denote the mean square for treatments and the mean square for error, respectively, from the analysis of variance, then we show in Appendix B that $MSE = U_1/2$ and $MST = [nU_2 + (1-n)U_1]/2$. The usual point estimate for ρ in the normal theory case is $n^{-1}(MST-MSE)/[n^{-1}(MST-MSE) + MSE]$ (Scheffe' 1959, Page 229). Using the above-mentioned relationships between MST, MSE, U_1 , and U_2 , it is easily seen that the normal theory and U-statistic approaches both lead to the same point estimate for ρ .

Consider now the statistic $T_k \equiv U_1/U_2$, and define a vector \underline{a} , such that

$$\underline{a}' = \left(\frac{\partial T_k}{\partial U_1}, \frac{\partial T_k}{\partial U_2} \right)$$

evaluated at the points $E(U_1) = 2\sigma_\epsilon^2$ and $E(U_2) = 2\sigma_\alpha^2 + 2\sigma_\epsilon^2$. Since

$$\frac{\partial T_k}{\partial U_1} = 1/U_2 \quad \text{and} \quad \frac{\partial T_k}{\partial U_2} = -U_1/U_2^2,$$

we have

$$(2.2.9) \quad \underline{a}' = ((2\sigma_a^2 + 2\sigma_\epsilon^2)^{-1}, (-2\sigma_\epsilon^2)/(2\sigma_a^2 + 2\sigma_\epsilon^2)^2).$$

Letting $\sigma_T^2 = \underline{a}' \underline{\Lambda} \underline{a}$, where $\underline{\Lambda}$ is as defined in (2.2.8), and using Theorem 2.1.2 and Theorem 14.6-2 from Bishop, Fienberg, and Holland (1975) concerning differentiable functions of vectors with a joint asymptotic normal distribution, we can state the following theorem.

Theorem 2.2.2. If $T_k = U_1/U_2$ and $\sigma_T^2 = \underline{a}' \underline{\Lambda} \underline{a}$, then

$$k^{1/2}[(1-T_k) - \rho]/\sigma_T \xrightarrow[k \rightarrow \infty]{d} N(0,1).$$

The quantity σ_T^2 depends on unknown parameters but Slutsky's Theorem (Serfling 1980, Page 19) assures us that T_k will still have an asymptotic normal distribution if we replace σ_T by a consistent estimate. Such an estimate is derived in Appendix C and is referred to here as $\hat{\sigma}_T$. Using this estimate, we can construct an asymptotic, $100(1-\zeta)\%$ confidence interval for ρ as

$$(2.2.10) \quad (1-U_1/U_2) \pm (Z_{\zeta/2}) (k^{-1/2}) \hat{\sigma}_T,$$

where $Z_{\zeta/2}$ denotes the $(1-\zeta/2)$ th percentile of a standard normal distribution.

The above procedure was derived before it was known that Arvesen (1969) had developed a similar procedure involving the jackknifing of U-statistics. Also, Arvesen and Schmitz (1970)

considered the specific problem of constructing an asymptotic confidence interval for the intraclass coefficient.

In their procedure Arvesen and Schmitz estimated $\beta = \ln(n\sigma_\alpha^2/\sigma_\varepsilon^2 + 1)$ by jackknifing the statistic $\hat{\beta}_k^0 = \ln(\text{MST}/\text{MSE})$ (note that MST/MSE can be written as a function of U-statistics: see Appendix B). The log transformation was used for variance stabilization which Arvesen and Schmitz showed, through simulation, was useful for moderate sample sizes.

The Arvesen-Schmitz procedure involves leaving out, one at a time, each of the vectors Z_i , and calculating $\hat{\beta}_{k-1}^i = \ln(\text{MST}/\text{MSE})$ using the remaining vectors as the data for a one-way design with $k - 1$ treatments. Using $\hat{\beta}_k^0$ as the estimate calculated using all k vectors, pseudo-estimates are formed as $\hat{\beta}_i = k\hat{\beta}_k^0 - (k-1)\hat{\beta}_{k-1}^i$. A point estimate is calculated as $\hat{\beta} = k^{-1}\sum_i \hat{\beta}_i$ and the standard deviation of the point estimate is estimated by $s_{\hat{\beta}} = [((k-1)^{-1}\sum_i (\hat{\beta}_i - \hat{\beta})^2)]^{1/2}$. Then, as in Tukey (1958), the distribution of the statistic,

$$t_{k-1} = k^{1/2}(\hat{\beta} - \beta)s_{\hat{\beta}}^{-1},$$

is approximated by a t distribution with $k - 1$ degrees of freedom.

If $t_{\zeta/2, k-1}$ is the $(1-\zeta/2)$ th percentile of a t distribution with $k - 1$ degrees of freedom, then an approximate $100(1-\zeta)\%$ confidence interval for β is

$$(\hat{\beta} - (t_{\zeta/2, k-1}s_{\hat{\beta}})^{-1/2}, \hat{\beta} + (t_{\zeta/2, k-1}s_{\hat{\beta}})^{-1/2}) \equiv (L, U).$$

Therefore, an approximate $100(1-\zeta)\%$ confidence interval for ρ is

$$(2.2.11) \quad ([\exp(L)-1]/[\exp(L)-1+n], [\exp(U)-1]/[\exp(U)-1+n]).$$

Another method of obtaining an asymptotic confidence interval for ρ can be derived using the fact that, for a two-dimensional vector \underline{W} , $\underline{W} \sim N_2(0, \underline{\Lambda})$ implies $\underline{W}' \underline{\Lambda}^{-1} \underline{W} \sim \chi^2_2$ where χ^2_2 is a chi-square random variable with two degrees of freedom (Serfling 1980, Page 128). Therefore, Theorem 2.2.1 implies that

$$(2.2.12) \quad k([U_1 - E(U_1), U_2 - E(U_2)]' \underline{\Lambda}^{-1} [U_1 - E(U_1), U_2 - E(U_2)])' \xrightarrow[k \rightarrow \infty]{} \chi^2_2.$$

Letting $D \equiv \text{Det}(\underline{\Lambda}) = \sigma_{11}\sigma_{22} - \sigma_{12}^2 > 0$ we obtain

$$\underline{\Lambda}^{-1} = D^{-1} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix}$$

and letting $X' = E(U_1) = 2\sigma_\epsilon^2$ and $Y' = E(U_2) = 2\sigma_\alpha^2 + 2\sigma_\epsilon^2$ we can rewrite (2.2.12) as

$$kD^{-1}[\sigma_{22}(X' - U_1)^2 + \sigma_{11}(Y' - U_2)^2 - 2\sigma_{12}(X' - U_1)(Y' - U_2)] \xrightarrow[k \rightarrow \infty]{} \chi^2_2.$$

Defining $\chi^2_{2\zeta}$ as the $(1-\zeta)$ th percentile of a χ^2_2 distribution and setting the above quadratic equation equal to $\chi^2_{2\zeta}$ we obtain

$$(2.2.13) \quad \sigma_{22}(X' - U_1)^2 + \sigma_{11}(Y' - U_2)^2 - 2\sigma_{12}(X' - U_1)(Y' - U_2) - D\chi^2_{2\zeta}k^{-1} = 0,$$

which is the equation of an ellipse such that the probability the point (U_1, U_2) is in the interior of the ellipse is approximately $1 - \zeta$.

Using the observed point, $(U_1, U_2) \equiv c$, as the center of the ellipse, we can form an asymptotic $100(1-\zeta)\%$ confidence interval for ρ in the following manner

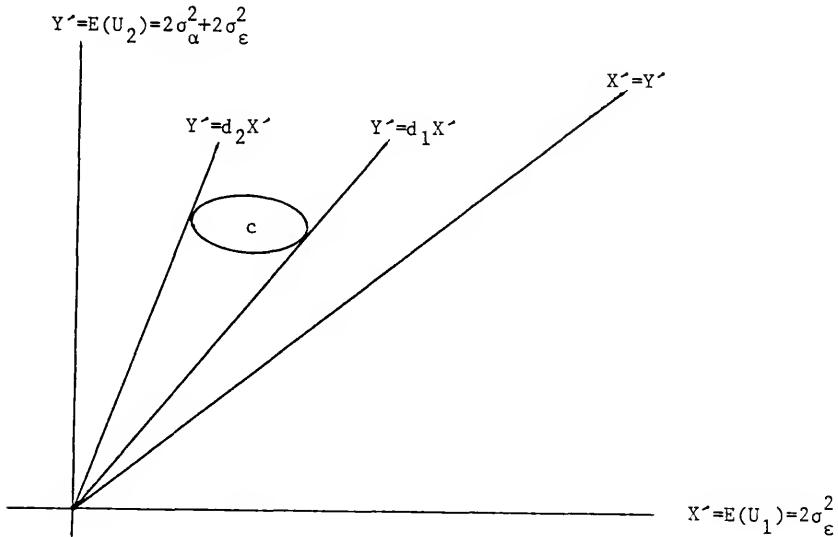


Figure 2.2.1

Let d_1 and d_2 ($d_1 < d_2$) be the slopes of the two lines that pass through the origin and intersect the ellipse in exactly one point (see Figure 2.2.1). Using $Y' = dX'$, or equivalently $d = (\sigma_\alpha^2 + \sigma_\epsilon^2)/\sigma_\epsilon^2$, we can form an asymptotic $100(1-\zeta)\%$ confidence interval for ρ as

$$(2.2.14) \quad (1 - d_1^{-1}, 1 - d_2^{-1}),$$

where the exact forms of d_1 and d_2 are given in Appendix D.

As we shall see in Chapter Four, this method of constructing a confidence interval for ρ is inferior to other available methods and therefore would not be recommended for use in practice.

CHAPTER THREE
CONFIDENCE INTERVALS USING MODIFIED ANSARI-BRADLEY STATISTICS

3.1 Model and Formation of Pseudo-Samples

Ansari and Bradley (1960) introduced a two-sample rank statistic that can be used to construct a confidence interval for the ratio of two scale parameters. Let X_1, X_2, \dots, X_n and Y_1, Y_2, \dots, Y_k be two independent samples of independent observations from populations with continuous distribution functions, $F(x)$ and $G(x)$ respectively, such that $F(x) = D(x/\delta_1)$ and $G(x) = D(x/\delta_2)$ for some distribution function $D(x)$. That is, δ_1 and δ_2 are scale parameters associated with the X 's and Y 's respectively. Define $\theta = \delta_2/\delta_1$, the ratio of the two scale parameters. Thus θX and Y have the same distribution.

The Ansari-Bradley statistic can be formulated in different ways. In the formulation we utilize, the combined sample of X 's and Y 's are ordered and the observations are ranked from the inside out as

$$N/2, \dots, 2, 1, 1, 2, \dots, N/2$$

if $N \leq n + k$ is even and as

$$(N-1)/2, \dots, 2, 1, 0, 1, 2, \dots, (N-1)/2$$

if N is odd. The Ansari-Bradley statistic is then defined as

$$W = \sum_i \text{Rank}(X_i).$$

The statistic W can be used as a test statistic for testing $H_0: \theta = 1$ versus one or two-sided alternatives. The distribution of W under the null hypothesis ($F(x) = G(x)$) is tabled for moderate values of n and k (Ansari and Bradley 1960). Bauer (1972) describes a method of inverting the test procedure to obtain a confidence interval for θ .

Using Theorem 1 of Chernoff and Savage (1958), Ansari and Bradley (1960) showed that $T_N = W/(nN)$ has an asymptotic normal distribution which, under the null hypothesis, has mean $1/4$ and variance $k(48nN)^{-1}$. However, the Ansari-Bradley statistic does not satisfy all the assumptions necessary for the application of Theorem 1 of Chernoff and Savage. An alternate proof of the asymptotic normality of T_N under the null hypothesis is given in Section 3.2. The alternate proof modifies the Chernoff and Savage proof so that it may be applied in the present situation.

Consider now the balanced one-way random effects model

$$z_{ij} = \mu + \alpha_i + \epsilon_{ij} \quad i = 1, 2, \dots, k, \quad j = 1, 2, \dots, n,$$

where μ is an unknown constant and the ϵ_{ij} and α_i are independent samples of independent observations from continuous distributions with distribution functions $F(x)$ and $G(x)$ and density functions $f(x)$ and $g(x)$ respectively. Also, assume there exist scale parameters, δ_1 and δ_2 , such that $F(x) = D(x/\delta_1)$ and $G(x) = D(x/\delta_2)$ where $D(x)$ is a continuous distribution function corresponding to a random variable with a distribution symmetric about zero. Thus, the ϵ_{ij} and the α_i are random variables with distributions symmetric about zero and they satisfy the assumptions needed for using the Ansari-Bradley statistic. Therefore, the ϵ_{ij}/δ_1 and the α_i/δ_2 have the same distribution.

The objective is to estimate the parameter $\gamma = \delta_2^2 / (\delta_1^2 + \delta_2^2)$ in order to assess whether the variability contributed by the treatments is large compared to the overall variability of the responses, i.e., to estimate the proportion of the variability in the responses attributable to the treatments.

Ordinarily in the two sample scale problem, θ is the parameter that would be of interest. However, in order to compare methods involving scale parameters to methods involving variances we instead look at the parameter γ which is a function of θ , namely $\gamma = \theta^2 / (\theta^2 + 1)$. Thus, γ and $\rho = \sigma_\alpha^2 / (\sigma_\alpha^2 + \sigma_\epsilon^2)$ are analogous parameters. In fact, $\gamma = \rho$ in those cases where both scale parameters and variances exist since $\delta_2 / \delta_1 = \sigma_\alpha / \sigma_\epsilon$. One advantage to a procedure that estimates γ is that an estimate of the desired quantity can be found in those cases where variances, and thus ρ , do not exist.

Ideally, we would like to have one sample consisting of the ϵ_{ij} and another consisting of the α_i and then use the Ansari-Bradley statistic to give us information about θ which could be transformed into a confidence interval for γ . However, knowing only the Z_{ij} , the individual ϵ_{ij} and α_i are not observable. What we can do is formulate a sample of size n that, as $n \rightarrow \infty$, essentially behaves like the ϵ_{ij} from treatment i and another sample of size k that, as n and $k \rightarrow \infty$, mimics the α_i .

The derivation of these two pseudo-samples follows. In the next section we will show that the asymptotic distribution of the Ansari-Bradley statistic calculated using these pseudo-samples is the same, when $\theta = 1$, as the asymptotic distribution of the Ansari-Bradley statistic calculated using the actual ϵ_{ij} from treatment i and the actual α_i .

We begin formation of the pseudo-samples by defining

$$\bar{z}_{ij} = n^{-1} \sum_j z_{ij}, \quad \bar{z}_{..} = (nk)^{-1} \sum_{ij} z_{ij}, \quad \bar{\epsilon}_{ij} = n^{-1} \sum_j \epsilon_{ij}, \quad \bar{\epsilon}_{..} = (nk)^{-1} \sum_{ij} \epsilon_{ij},$$

and $\bar{\alpha} = k^{-1} \sum_i \alpha_i$. The two pseudo-samples we obtain are

$$\begin{array}{ll}
 \frac{1}{\overline{Z}_{i1} - \overline{Z}_{i.}} = \varepsilon_{i1} - \overline{\varepsilon}_{i.} & \frac{2}{\overline{Z}_{1.} - \overline{Z}_{..}} = \alpha_1 + \overline{\varepsilon}_{1.} - \overline{\alpha} - \overline{\varepsilon}_{..} \\
 x_1 = z_{i1} - \overline{z}_{i.} & y_1 = \overline{z}_{1.} - \overline{z}_{..} \\
 x_2 = z_{i2} - \overline{z}_{i.} & y_2 = \overline{z}_{2.} - \overline{z}_{..} \\
 \cdot & \cdot \\
 (3.1.1) & \cdot \\
 \cdot & \cdot \\
 \cdot & \cdot \\
 \cdot & \cdot \\
 y_k = \overline{z}_{k.} - \overline{z}_{..} & = \alpha_k + \overline{\varepsilon}_{k.} - \overline{\alpha} - \overline{\varepsilon}_{..} \\
 x_n = z_{in} - \overline{z}_{i.} & = \varepsilon_{in} - \overline{\varepsilon}_{i.}
 \end{array}$$

Under the assumptions of Theorem 3.2.1, as n and $k \rightarrow \infty$, $\bar{\varepsilon}_{\cdot j}$, $\bar{\alpha}_j$, and $\bar{\varepsilon}_{i \cdot}$, for $1 \leq i \leq n$, converge in probability to zero. Therefore, for large N , we would expect the X_j and the Y_i to behave like random samples from $F(x)$ and $G(x)$ respectively.

Throughout this chapter we assume that n and k both tend to infinity in such a way that $\lambda_N = n/N$ always satisfies the condition that $\lambda_0 < \lambda_N < 1 - \lambda_0$ for $0 < \lambda_0 < 1/2$. Obviously, $N = n + k$ will therefore tend to infinity. To facilitate discussions, we will simply say that $N \rightarrow \infty$.

Recall that δ_1 is a scale parameter for the ε_{ij} and δ_2 is a scale parameter for the α_i . Let $F^*(x)$ and $G^*(x)$ be the distribution functions for the X_j and Y_i respectively. In an asymptotic sense we can think of δ_1 as being a scale parameter for the X_j and δ_2 as being a scale parameter for the Y_i since

$$\begin{aligned} F^*(x) &= P(X_j < x) = P(\varepsilon_{ij} - \bar{\varepsilon}_{i..} < x) \\ &\xrightarrow[N \rightarrow \infty]{} P(\varepsilon_{ij} < x) = F(x) = D(x/\delta_1) \end{aligned}$$

and

$$\begin{aligned} G^*(x) &= P(Y_i < x) = P(\alpha_i + \bar{\varepsilon}_{i..} - \bar{\alpha} - \bar{\varepsilon}_{..} < x) \\ &\xrightarrow[N \rightarrow \infty]{} P(\alpha_i < x) = G(x) = D(x/\delta_2). \end{aligned}$$

These asymptotic equivalences can be justified by noting again that the means converge in probability to zero and using Slutsky's Theorem (Serfling 1980, Page 19).

Samples with the same asymptotic properties as those in (3.1.1) can be obtained in other ways. One approach is to use medians rather than means. Define $\hat{z}_i = \text{median of } (z_{i1}, z_{i2}, \dots, z_{in})$, $\hat{z} = \text{median of } (\hat{z}_1, \hat{z}_2, \dots, \hat{z}_k)$, $\hat{\varepsilon}_i = \text{median of } (\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{in})$, and $\hat{\alpha} = \text{median of } (\alpha_1 + \hat{\varepsilon}_1, \alpha_2 + \hat{\varepsilon}_2, \dots, \alpha_k + \hat{\varepsilon}_k)$. We could then obtain the pseudo-samples

$$\begin{array}{ll} \frac{1}{\cdot} & \frac{2}{\cdot} \\ X'_1 = z_{i1} - \hat{z}_i & Y'_1 = \hat{z}_1 - \hat{z} = \alpha_1 + \hat{\varepsilon}_1 - \hat{\alpha} \\ X'_2 = z_{i2} - \hat{z}_i & Y'_2 = \hat{z}_2 - \hat{z} = \alpha_2 + \hat{\varepsilon}_2 - \hat{\alpha} \\ \cdot & \cdot \\ \cdot & \cdot \\ X'_k = z_{in} - \hat{z}_i & Y'_k = \hat{z}_k - \hat{z} = \alpha_k + \hat{\varepsilon}_k - \hat{\alpha} \\ X'_n = z_{in} - \hat{z}_i & \end{array} \quad (3.1.2)$$

Using reasoning similar to that used with the samples in (3.1.1), under the assumptions of Theorem 3.2.2, we conclude that these pseudo-

samples would be asymptotically equivalent to random samples from $F(x)$ and $G(x)$ respectively.

Other methods of obtaining the two pseudo-samples could be used as long as they provided samples with the correct asymptotic properties, the estimates involved converge to zero, and the estimates satisfy other criteria that will be examined in Section 3.2.

In the next section, we turn our attention to the asymptotic distribution of the Ansari-Bradley statistic calculated using the pseudo-samples. We show that, when $\theta = 1$, this distribution is equivalent to the asymptotic distribution of the Ansari-Bradley statistic calculated using the actual ε_{ij} and α_i .

3.2 Asymptotic Distribution of the Ansari-Bradley Statistic Using Pseudo-Samples

Consider the pseudo-samples described in (3.1.1). For simplicity, but with no loss of generality, we will assume that we are using the ε_{ij} from treatment one. Let $\underline{\varepsilon}' = (\varepsilon_{11}, \varepsilon_{12}, \dots, \varepsilon_{1n})$, $\underline{\alpha}' = (\alpha_1, \alpha_2, \dots, \alpha_k)$, $\underline{X}' = (X_1, X_2, \dots, X_n)$, $\underline{Y}' = (Y_1, Y_2, \dots, Y_k)$, and $W(\underline{\varepsilon}, \underline{\alpha})$ denote the Ansari-Bradley statistic calculated using the samples $\underline{\varepsilon}$ and $\underline{\alpha}$. We now derive an expression for $W(\underline{\varepsilon}, \underline{\alpha})$ that is similar to a Chernoff and Savage (1958) expansion of a statistic. We do not use a direct application of the Chernoff and Savage procedure because the Ansari-Bradley statistic does not satisfy all the assumptions necessary for implementing the Chernoff and Savage expansion. However, using an alternative expansion that produces a similar expression to that obtained by Chernoff and Savage allows us to make use of some of their results.

For any event A, let $I(A)$ be 1 if event A occurs and 0 if event A does not occur. We then define empirical distribution functions for the ε_{1j} and the α_i as $F_n(x) = n^{-1} \sum_j I(\varepsilon_{1j} < x)$ and $G_k(x) = k^{-1} \sum_i I(\alpha_i < x)$.

With λ_N as described in Section 3.1, we then define the combined sample empirical distribution function as $H_N(x) = \lambda_N F_n(x) + (1-\lambda_N) G_k(x)$ and let the combined population distribution function be denoted

$$(3.2.1) \quad H(x) = \lambda_N F(x) + (1-\lambda_N) G(x).$$

We define a function $J_N[H_N(x)]$ to be

$$(3.2.2) \quad J_N[H_N(x)] = \begin{cases} (2N)^{-1} + |1/2 + (2N)^{-1} - H_N(x)| & \text{if } N \text{ is even} \\ |1/2 + (2N)^{-1} - H_N(x)| & \text{if } N \text{ is odd} \end{cases}$$

and a function $J[H(x)]$ as

$$(3.2.3) \quad J[H(x)] = |1/2 - H(x)|.$$

We also let

$$(3.2.4) \quad J'[H(x)] = \begin{cases} -1 & \text{if } H(x) < 1/2 \\ 1 & \text{if } H(x) > 1/2 \end{cases}$$

and note that $J'[H(x)]$ is the derivative of $J[H(x)]$ with respect to $H(x)$ at all points except $H(x) = 1/2$ (where the derivative is not defined).

We make $J'(1/2) = -1$ by definition so $J'[H(x)]$ will be defined everywhere.

Let

$$(3.2.5) \quad T_N(\underline{\varepsilon}, \underline{\alpha}) = (nN)^{-1} W(\underline{\varepsilon}, \underline{\alpha}) = \int J_N[H_N(x)] dF_n(x)$$

be an alternative representation of the Ansari-Bradley statistic and let

$$(3.2.6) \quad A = \int J[H(x)]dF(x),$$

$$(3.2.7) \quad B_{1N} = \int J[H(x)]d[F_n(x)-F(x)],$$

$$(3.2.8) \quad B_{2N} = \int [H_N(x)-H(x)]J'[H(x)]dF(x),$$

$$(3.2.9.a) \quad C_{1N} = \lambda_N \int [F_n(x)-F(x)]J'[H(x)]d[F_n(x)-F(x)],$$

$$(3.2.9.b) \quad C_{2N} = (1-\lambda_N) \int [G_k(x)-G(x)]J'[H(x)]d[F_n(x)-F(x)],$$

$$(3.2.9.c) \quad C_{3N} = \int (J_N[H_N(x)] - J[H_N(x)])dF_n(x).$$

Then, by adding and subtracting appropriate quantities,

$$\begin{aligned} T_N(\xi, \underline{\alpha}) &= \int J_N[H_N(x)]dF_n(x) \\ &= \int J[H(x)]dF(x) + \int J[H(x)]d[F_n(x)-F(x)] \\ &\quad + \int (J_N[H_N(x)] - J[H_N(x)])dF_n(x) \\ &\quad + \int (J[H_N(x)] - J[H(x)])dF_n(x) \\ &= A + B_{1N} + C_{3N} + \int (J[H_N(x)] - J[H(x)])dF_n(x). \end{aligned}$$

Now, using (3.2.3), $J[H_N(x)] - J[H(x)]$ is equal to

$$\left\{ \begin{array}{ll} H(x) - H_N(x) & \text{if } 0 < H_N(x) < 1/2 \text{ and } 0 < H(x) < 1/2 \\ H_N(x) - H(x) & \text{if } 1/2 < H_N(x) < 1 \text{ and } 1/2 < H(x) < 1 \\ 1 - H_N(x) - H(x) & \text{if } 0 < H_N(x) < 1/2 \text{ and } 1/2 < H(x) < 1 \\ H_N(x) + H(x) - 1 & \text{if } 1/2 < H_N(x) < 1 \text{ and } 0 < H(x) < 1/2. \end{array} \right.$$

Since we assume that $F(x)$ and $G(x)$ are such that $F(0) = 1/2 = G(0)$, it follows that $H(0) = 1/2$. Thus, using (3.2.4), $J[H_N(x)] - J[H(x)]$ is equal to

$$\left\{ \begin{array}{ll} [H_N(x) - H(x)]J'[H(x)] + 0 & \text{if } 0 < H_N(x) < 1/2 \text{ and } x < 0 \\ [H_N(x) - H(x)]J'[H(x)] + 0 & \text{if } 1/2 < H_N(x) < 1 \text{ and } x > 0 \\ [H_N(x) - H(x)]J'[H(x)] + [1 - 2H_N(x)]J'[H(x)] & \text{if } 0 < H_N(x) < 1/2 \text{ and } x > 0 \\ [H_N(x) - H(x)]J'[H(x)] + [1 - 2H_N(x)]J'[H(x)] & \text{if } 1/2 < H_N(x) < 1 \text{ and } x < 0. \end{array} \right.$$

We define

$$K_N(x) = \left\{ \begin{array}{ll} 0 & \text{if } 0 < H_N(x) < 1/2 \text{ and } x < 0 \\ 0 & \text{if } 1/2 < H_N(x) < 1 \text{ and } x > 0 \\ [1 - 2H_N(x)]J'[H(x)] & \text{if } 0 < H_N(x) < 1/2 \text{ and } x > 0 \\ [1 - 2H_N(x)]J'[H(x)] & \text{if } 1/2 < H_N(x) < 1 \text{ and } x < 0 \end{array} \right.$$

and

$$(3.2.9.d) \quad C_{4N} = \int K_N(x) dF_N(x).$$

Then $J[H_N(x)] - J[H(x)] = [H_N(x) - H(x)]J'[H(x)] + K_N(x)$ and therefore

$$\begin{aligned} T_N(\underline{\xi}, \underline{\alpha}) &= A + B_{1N} + C_{3N} \int [H_N(x) - H(x)]J'[H(x)]dF_N(x) + C_{4N} \\ &= A + B_{1N} + C_{3N} + \int [H_N(x) - H(x)]J'[H(x)]dF(x) \\ &\quad + \int [H_N(x) - H(x)]J'[H(x)]d[F_N(x) - F(x)] + C_{4N} \\ &= A + B_{1N} + B_{2N} + C_{3N} + \lambda_N \int [F_N(x) - F(x)]J'[H(x)]d[F_N(x) - F(x)] \\ &\quad + (1 - \lambda_N) \int [G_k(x) - G(x)]J'[H(x)]d[F_N(x) - F(x)] + C_{4N} \\ (3.2.10) \quad &= A + B_{1N} + B_{2N} + C_{1N} + C_{2N} + C_{3N} + C_{4N}. \end{aligned}$$

The terms C_{1N} through C_{4N} are shown to be of order $o_p(N^{-1/2})$ in Appendix E. Since A , B_{1N} , and B_{2N} are equal to or analogous to corresponding terms in a Chernoff and Savage (1958) expansion, we see that $T_N(\underline{\xi}, \underline{\alpha})$ has an asymptotic normal distribution with mean

$$\mu_N = \int J[H(x)]dF(x)$$

and variance

$$\begin{aligned} N\sigma_N^2 &= 2(1-\lambda_N) \left(\int_{-\infty}^{\infty} \int_{-\infty}^y G(x)[1-G(y)]J'[H(x)]J'[H(y)]dF(x)dF(y) \right. \\ &\quad \left. + (1-\lambda_N)\lambda_N^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^y F(x)[1-F(y)]J'[H(x)]J'[H(y)]dG(x)dG(y) \right). \end{aligned}$$

Under $H_0: \theta = 1$, implying $\delta_1 = \delta_2$ and therefore $F(x) = G(x) = H(x)$, it can be shown that $\mu_N = 1/4$ and $\sigma_N^2 = k(48nN)^{-1}$.

To prove that $W(\underline{X}, \underline{Y})$, where \underline{X} and \underline{Y} are as given in (3.1.1), has the same asymptotic distribution as $W(\underline{\varepsilon}, \underline{g})$, when $\theta = 1$, we look at

$$(3.2.11) \quad T_N(\underline{X}, \underline{Y}) = (nN)^{-1} W(\underline{X}, \underline{Y})$$

and show that it has the same asymptotic distribution as $T_N(\underline{\varepsilon}, \underline{g})$.

Remembering that $F(x)$ and $G(x)$ are the true distribution functions for the ε_{1j} and the α_i respectively and that $F_n(x)$ and $G_k(x)$ are the corresponding empirical distribution functions, we define

$$(3.2.12) \quad F_n^*(x) = n^{-1} \sum_j I(X_j < x),$$

$$(3.2.13) \quad G_k^*(x) = k^{-1} \sum_i I(Y_i < x),$$

$$(3.2.14) \quad H_N^*(x) = \lambda_N F_n^*(x) + (1-\lambda_N) G_k^*(x),$$

$$(3.2.15) \quad A^* = \int J[H(x)]dF(x),$$

$$(3.2.16) \quad B_{1N}^* = \int J[H(x)]d[F_n^*(x)-F(x)],$$

$$(3.2.17) \quad B_{2N}^* = \int [H_N^*(x)-H(x)]J'[H(x)]dF(x),$$

$$(3.2.18.a) \quad C_{1N}^* = \lambda_N \int [F_n^*(x)-F(x)]J'[H(x)]d[F_n^*(x)-F(x)],$$

$$(3.2.18.b) \quad C_{2N}^* = (1-\lambda_N) \int [G_k^*(x)-G(x)]J'[H(x)]d[F_n^*(x)-F(x)],$$

$$(3.2.18.c) \quad C_{3N}^* = \int (J_N[H_N^*(x)] - J[H_N^*(x)]) dF_n^*(x),$$

$$(3.2.18.d) \quad C_{4N}^* = \int K_N^*(x) dF_n^*(x),$$

where

$$K_N^*(x) = \begin{cases} 0 & \text{if } 0 < H_N^*(x) < 1/2 \text{ and } x < 0 \\ 0 & \text{if } 1/2 < H_N^*(x) < 1 \text{ and } x > 0 \\ [1 - 2H_N^*(x)] J'[H(x)] & \text{if } 0 < H_N^*(x) < 1/2 \text{ and } x > 0 \\ [1 - 2H_N^*(x)] J'[H(x)] & \text{if } 1/2 < H_N^*(x) < 1 \text{ and } x < 0 \end{cases}$$

and $H(x)$, J_N , J , and J' are as defined in (3.2.1) through (3.2.4).

Expanding $T_N(\underline{x}, \underline{y}) = \int J_N[H_N^*(x)] dF_n^*(x)$ in the same manner as we expanded $T_N(\underline{s}, \underline{g})$ produces

$$(3.2.19) \quad T_N(\underline{x}, \underline{y}) = A^* + B_{1N}^* + B_{2N}^* + C_{1N}^* + C_{2N}^* + C_{3N}^* + C_{4N}^*.$$

Recall that $F(x)$, $G(x)$, and thus $H(x)$, are assumed to represent distributions symmetric about zero. This implies that $J[H(0)] = J(1/2) = 0$. Therefore, defining

$$(3.2.20) \quad \hat{B}(x) = \int_0^x J'[H(y)] dF(y)$$

and

$$(3.2.21) \quad B(x) = \int_0^x J'[H(y)] dG(y),$$

B_{1N}^* can be written as

$$\begin{aligned} & \int (\int_0^x J'[H(y)] dH(y)) d[F_n^*(x) - F(x)] \\ &= \int (\int_0^x J'[H(y)] d[\lambda_N F(y) + (1-\lambda_N) G(y)]) d[F_n^*(x) - F(x)] \\ &= \lambda_N \int \hat{B}(x) d[F_n^*(x) - F(x)] + (1-\lambda_N) \int B(x) d[F_n^*(x) - F(x)]. \end{aligned}$$

Integrating B_{2N}^* by parts produces

$$\begin{aligned} & [H_N^*(x) - H(x)] \hat{B}(x) \Big|_{-\infty}^{\infty} - \int \hat{B}(x) d[H_N^*(x) - H(x)] \\ & = -(\lambda_N \int \hat{B}(x) d[F_N^*(x) - F(x)] + (1 - \lambda_N) \int \hat{B}(x) d[G_k^*(x) - G(x)]). \end{aligned}$$

Thus, $B_{1N}^* + B_{2N}^*$ may be expressed as

$$\begin{aligned} (3.2.22) \quad & (1 - \lambda_N) (\int B(x) d[F_N^*(x) - F(x)] - \int \hat{B}(x) d[G_k^*(x) - G(x)]) \\ & = (1 - \lambda_N) (n^{-1} \sum_j B(\epsilon_{1j}) - \bar{\epsilon}_{1.}) - E[B(\epsilon_{11})] \\ & \quad - k^{-1} \sum_i \hat{B}(\alpha_i + \bar{\epsilon}_{1.} - \bar{\alpha} - \bar{\epsilon}_{..}) - E[\hat{B}(\alpha_1)]. \end{aligned}$$

In the same manner it can be shown that $B_{1N} + B_{2N}$ (given in (3.2.7) and (3.2.8)) can be written as

$$(3.2.23) \quad (1 - \lambda_N) (n^{-1} \sum_j B(\epsilon_{1j}) - E[B(\epsilon_{11})]) - k^{-1} \sum_i \hat{B}(\alpha_i) - E[\hat{B}(\alpha_1)].$$

Recalling the form of $J'[H(x)]$ given in (3.2.4) and the definitions of $\hat{B}(x)$ and $B(x)$ given in (3.2.20) and (3.2.21), we see that

$$(3.2.24) \quad \hat{B}(x) = \begin{cases} 1/2 - F(x) & \text{if } x \leq 0 \\ F(x) - 1/2 & \text{if } x > 0 \end{cases}$$

and

$$(3.2.25) \quad B(x) = \begin{cases} 1/2 - G(x) & \text{if } x \leq 0 \\ G(x) - 1/2 & \text{if } x > 0. \end{cases}$$

We then define

$$(3.2.26) \quad \hat{B}'(x) = \begin{cases} -f(x) & \text{if } x \leq 0 \\ f(x) & \text{if } x > 0 \end{cases}$$

and

$$(3.2.27) \quad B'(x) = \begin{cases} -g(x) & \text{if } x < 0 \\ g(x) & \text{if } x > 0, \end{cases}$$

noting that $\hat{B}'(x)$ and $B'(x)$ are the respective derivatives of $\hat{E}(x)$ and $E(x)$, except at $x = 0$. We make $\hat{B}'(0) = -f(0)$ and $B'(0) = -g(0)$ by definition so the functions will be defined everywhere.

The proof that the asymptotic distribution of $T_N(\underline{x}, \underline{y})$ is the same as that of $T_N(\underline{\epsilon}, \underline{g})$, when $\theta = 1$, is given in the following theorem.

Theorem 3.2.1. Using the model and assumptions described in Section 3.1 and the pseudo-samples given in (3.1.1), if

- (i) $\theta = \delta_2/\delta_1 = 1$ (WLOG assume $\delta_1 = \delta_2 = 1$),
- (ii) $F'(x) = f(x)$ and $|F''(x)| = |f''(x)|$ are continuous and bounded by constants B_1 and B_2 respectively,
- (iii) $f(0) > 0$,
- (iv) $\int x^2 f(x) dx < \infty$,

then

$$(3.2.28) \quad N^{1/2} [T_N(\underline{x}, \underline{y}) - 1/4] (48nk^{-1})^{1/2} \xrightarrow[N \rightarrow \infty]{d} N(0, 1).$$

Proof: First note (i) implies $F(x) = G(x) = H(x)$ and (iv) implies $N^{1/2}\bar{\epsilon}_\alpha = o_p(1)$. Assumption (iv) also implies that for every i , $N^{1/2}\bar{\epsilon}_{i.} = o_p(1)$ and $E[(N^{1/2}\bar{\epsilon}_{i.})^2]$ is uniformly bounded. Also note that the Glivenko-Cantelli Theorem (Serfling 1980, Page 61) states that $\sup_x |F_n(x) - F(x)| = o_p(1)$. We begin the proof of the theorem by proving two lemmas.

Lemma 3.2.1. Under the assumptions of Theorem 3.2.1,

$$n^{-1} N^{1/2} \sum_j [B(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) - B(\varepsilon_{1j})] = o_p(1).$$

Proof: Recalling the form of $B(x)$ from (3.2.25) we see that $B(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) - B(\varepsilon_{1j})$ is equal to

$$\begin{cases} G(\varepsilon_{1j}) - G(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) & \text{if } \varepsilon_{1j} < 0 \text{ and } \varepsilon_{1j} - \bar{\varepsilon}_{1.} < 0 \\ G(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) - G(\varepsilon_{1j}) & \text{if } \varepsilon_{1j} > 0 \text{ and } \varepsilon_{1j} - \bar{\varepsilon}_{1.} > 0 \\ 1 - G(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) - G(\varepsilon_{1j}) & \text{if } \varepsilon_{1j} > 0 \text{ and } \varepsilon_{1j} - \bar{\varepsilon}_{1.} < 0 \\ G(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) + G(\varepsilon_{1j}) - 1 & \text{if } \varepsilon_{1j} < 0 \text{ and } \varepsilon_{1j} - \bar{\varepsilon}_{1.} > 0, \end{cases}$$

which can also be written as

$$\begin{cases} -[G(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) - G(\varepsilon_{1j})] + 0 & \text{if } \varepsilon_{1j} < 0 \text{ and } \varepsilon_{1j} - \bar{\varepsilon}_{1.} < 0 \\ [G(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) - G(\varepsilon_{1j})] + 0 & \text{if } \varepsilon_{1j} > 0 \text{ and } \varepsilon_{1j} - \bar{\varepsilon}_{1.} > 0 \\ [G(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) - G(\varepsilon_{1j})] + [1 - 2G(\varepsilon_{1j} - \bar{\varepsilon}_{1.})] & \text{if } \varepsilon_{1j} > 0 \text{ and } \varepsilon_{1j} - \bar{\varepsilon}_{1.} < 0 \\ -[G(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) - G(\varepsilon_{1j})] + [2G(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) - 1] & \text{if } \varepsilon_{1j} < 0 \text{ and } \varepsilon_{1j} - \bar{\varepsilon}_{1.} > 0. \end{cases}$$

Defining

$$(3.2.29) \quad B_N(X_j) = \begin{cases} 0 & \text{if } \varepsilon_{1j} < 0 \text{ and } \varepsilon_{1j} - \bar{\varepsilon}_{1.} < 0 \\ 0 & \text{if } \varepsilon_{1j} > 0 \text{ and } \varepsilon_{1j} - \bar{\varepsilon}_{1.} > 0 \\ 1 - 2G(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) & \text{if } \varepsilon_{1j} > 0 \text{ and } \varepsilon_{1j} - \bar{\varepsilon}_{1.} < 0 \\ 2G(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) - 1 & \text{if } \varepsilon_{1j} < 0 \text{ and } \varepsilon_{1j} - \bar{\varepsilon}_{1.} > 0 \end{cases}$$

and noting that

$$G(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) - G(\varepsilon_{1j}) = (-\bar{\varepsilon}_{1.})g(\varepsilon_{1j}) + (1/2\bar{\varepsilon}_{1.}^2)g'(\varepsilon_{1j} + \tau_j),$$

where τ_j is between 0 and $-\bar{\varepsilon}_{1.}$, we use the form of $B'(x)$ (3.2.27) to obtain

$$\begin{aligned}
 & n^{-1} N^{1/2} \sum_j [B(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) - B(\varepsilon_{1j})] \\
 & = -(N^{1/2} \bar{\varepsilon}_{1.}) n^{-1} \sum_j B'(\varepsilon_{1j}) + (1/2 \bar{\varepsilon}_{1.}) (N^{1/2} \bar{\varepsilon}_{1.}) n^{-1} \sum_j J'[G(\varepsilon_{1j})] g'(\varepsilon_{1j} + \tau_j) \\
 (3.2.30) \quad & + n^{-1} N^{1/2} \sum_j B_N(X_j).
 \end{aligned}$$

Now, $E[B'(\varepsilon_{1j})] = \int_{-\infty}^0 -f(x)dF(x) + \int_0^\infty f(x)dF(x) = 0$ since $f(x)$ is symmetric about zero. Also, $E[B'^2(\varepsilon_{1j})] = \int f^2(x)dF(x)$ is bounded since $f(x)$ is bounded. Therefore, since the ε_{1j} are independent, we apply the Markov inequality (Chow and Teicher 1978, Page 88) to obtain

$n^{-1} \sum_j B'(\varepsilon_{1j}) = o_p(1)$. The assumptions of the theorem imply $\bar{\varepsilon}_{1.} = o_p(1)$, $N^{1/2} \bar{\varepsilon}_{1.} = o_p(1)$, and $|J'[G(\varepsilon_{1j})]g'(\varepsilon_{1j} + \tau_j)| \leq B_2$. Thus, the sum of the first two terms of (3.2.30) is $o_p(1)o_p(1) + o_p(1)o_p(1)o_p(1) = o_p(1)$. To complete the proof of the lemma we must show $n^{-1} N^{1/2} \sum_j B_N(X_j) = o_p(1)$.

Using (3.2.29) we see that $n^{-1} N^{1/2} \sum_j B_N(X_j)$ is equal to

$$\begin{aligned}
 & n^{-1} N^{1/2} \sum_j ([1 - 2G(\varepsilon_{1j} - \bar{\varepsilon}_{1.})] I(\varepsilon_{1j} > 0) I(\varepsilon_{1j} - \bar{\varepsilon}_{1.} < 0) \\
 & \quad + [2G(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) - 1] I(\varepsilon_{1j} < 0) I(\varepsilon_{1j} - \bar{\varepsilon}_{1.} > 0)) \\
 & = 2n^{-1} N^{1/2} \sum_j ([G(0) - G(\varepsilon_{1j} - \bar{\varepsilon}_{1.})] I(0 < \varepsilon_{1j} < \bar{\varepsilon}_{1.}) \\
 & \quad + [G(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) - G(0)] I(\bar{\varepsilon}_{1.} < \varepsilon_{1j} < 0)) \\
 & \leq 2n^{-1} N^{1/2} \sum_j ([G(\varepsilon_{1j}) - G(\varepsilon_{1j} - \bar{\varepsilon}_{1.})] I(0 < \varepsilon_{1j} < \bar{\varepsilon}_{1.}) \\
 & \quad + [G(\varepsilon_{1j} - \bar{\varepsilon}_{1.}) - G(\varepsilon_{1j})] I(\bar{\varepsilon}_{1.} < \varepsilon_{1j} < 0)) \\
 & = 2n^{-1} N^{1/2} \sum_j (|(-\bar{\varepsilon}_{1.})g(\varepsilon_{1j} + \tau'_j)| [I(0 < \varepsilon_{1j} < \bar{\varepsilon}_{1.}) + I(\bar{\varepsilon}_{1.} < \varepsilon_{1j} < 0)])
 \end{aligned}$$

where τ'_j is between 0 and $-\bar{\varepsilon}_{1.}$

$$\begin{aligned}
& \leq 2B_1 N^{1/2} |\bar{\varepsilon}_{1.}| n^{-1} \sum_j [I(0 < \varepsilon_{1j} < \bar{\varepsilon}_{1.}) + I(\bar{\varepsilon}_{1.} < \varepsilon_{1j} < 0)] \\
& = 2B_1 N^{1/2} |\bar{\varepsilon}_{1.}| |F_n(0) - F_n(\bar{\varepsilon}_{1.})| \\
& \leq [2B_1 N^{1/2} |\bar{\varepsilon}_{1.}|] [|F_n(0) - F(0)| + |F(0) - F(\bar{\varepsilon}_{1.})| + |F(\bar{\varepsilon}_{1.}) - F_n(\bar{\varepsilon}_{1.})|] \\
& = o_p(1)[o_p(1) + o_p(1) + o_p(1)] \\
& = o_p(1),
\end{aligned}$$

using the Glivenko-Cantelli Theorem, the continuity of $F(x)$, and the fact that $\bar{\varepsilon}_{1.} = o_p(1)$. This completes the proof of the lemma.

Lemma 3.2.2. Under the assumptions of Theorem 3.2.1,

$$k^{-1} N^{1/2} \sum_i [\hat{B}(\alpha_i + \bar{\varepsilon}_{1.} - \bar{\alpha} - \bar{\varepsilon}_{..}) - \hat{B}(\alpha_i)] = o_p(1).$$

Proof: We begin by recalling the forms of $\hat{B}(x)$ and $\hat{B}'(x)$ given in (3.2.24) and (3.2.26). Proceeding as in the proof of Lemma 3.2.1, we define

$$(3.2.31) \quad \hat{B}_N(Y_i) = \begin{cases} 0 & \text{if } \alpha_i < 0 \text{ and } \alpha_i + \bar{\varepsilon}_{1.} - \bar{\alpha} - \bar{\varepsilon}_{..} < 0 \\ 0 & \text{if } \alpha_i > 0 \text{ and } \alpha_i + \bar{\varepsilon}_{1.} - \bar{\alpha} - \bar{\varepsilon}_{..} > 0 \\ 1 - 2F(\alpha_i + \bar{\varepsilon}_{1.} - \bar{\alpha} - \bar{\varepsilon}_{..}) & \text{if } \alpha_i > 0 \text{ and } \alpha_i + \bar{\varepsilon}_{1.} - \bar{\alpha} - \bar{\varepsilon}_{..} < 0 \\ 2F(\alpha_i + \bar{\varepsilon}_{1.} - \bar{\alpha} - \bar{\varepsilon}_{..}) - 1 & \text{if } \alpha_i < 0 \text{ and } \alpha_i + \bar{\varepsilon}_{1.} - \bar{\alpha} - \bar{\varepsilon}_{..} > 0 \end{cases}$$

and obtain

$$\begin{aligned}
& k^{-1} N^{1/2} \sum_i [\hat{B}(\alpha_i + \bar{\varepsilon}_{1.} - \bar{\alpha} - \bar{\varepsilon}_{..}) - \hat{B}(\alpha_i)] \\
(3.2.32) \quad & = -N^{1/2} (\bar{\alpha} + \bar{\varepsilon}_{..}) k^{-1} \sum_i \hat{B}'(\alpha_i) + k^{-1} \sum_i (N^{1/2} \bar{\varepsilon}_{1.}) \hat{B}'(\alpha_i) \\
& + N^{1/2} (2k)^{-1} \sum_i (\bar{\varepsilon}_{1.} - \bar{\alpha} - \bar{\varepsilon}_{..})^2 J' [F(\alpha_i)] f'(\alpha_i + \tau_i) \\
& + k^{-1} N^{1/2} \sum_i \hat{B}_N(Y_i),
\end{aligned}$$

where τ_i is between 0 and $\bar{\epsilon}_{i.} - \bar{\alpha} - \bar{\epsilon}_{..}$. To prove the lemma we show that each of the four terms in (3.2.32) is $o_p(1)$.

Since $N^{1/2}\bar{\alpha} = o_p(1)$ and $N^{1/2}\bar{\epsilon}_{..} = o_p(1)$, it follows that $N^{1/2}(\bar{\alpha} + \bar{\epsilon}_{..}) = o_p(1)$ and therefore the first term in (3.2.32) is seen to be $o_p(1)$ following the same argument used to show $(N^{1/2}\bar{\epsilon}_{i.})_{n=1}^N \sum_j B'(\epsilon_{1j})$ was $o_p(1)$ in the proof of Lemma 3.2.1.

In the second term of (3.2.32) let $A_N = k^{-1} \sum_i (N^{1/2}\bar{\epsilon}_{i.}) \hat{B}'(\alpha_i)$. Since the $\bar{\epsilon}_{i.}$ and the $\hat{B}'(\alpha_i)$ are independent with zero means, we see that $E(A_N) = 0$. We also note that $E(A_N^2) = Nk^{-1} E[(\bar{\epsilon}_{i.})^2 \hat{B}'^2(\alpha_i)]$. The assumptions of the theorem guarantee that $|\hat{B}'(\alpha_i)|$ is bounded and that $E(\bar{\epsilon}_{i.}^2) = o_p(1)$. It then follows that $E(A_N^2) = o(1)$ and thus, using the Markov inequality, that $A_N = o_p(1)$.

We now turn attention to the third term of (3.2.32). From assumption (ii) and the definition of J' , we see that

$$|J'[F(\alpha_i)]f'(\alpha_i + \tau_i)| < B_2. \quad \text{We can then write}$$

$$\begin{aligned} & |N^{1/2}(2k)^{-1} \sum_i (\bar{\epsilon}_{i.} - \bar{\alpha} - \bar{\epsilon}_{..})^2 J'[F(\alpha_i)]f'(\alpha_i + \tau_i)| \\ & \leq B_2 N^{1/2} (2k)^{-1} \sum_i (\bar{\epsilon}_{i.} - \bar{\alpha} - \bar{\epsilon}_{..})^2. \end{aligned}$$

Expanding the upper bound we obtain

$$(3.2.33) \quad \begin{aligned} & B_2 / 2 [N^{1/2}(\bar{\alpha} + \bar{\epsilon}_{..})(\bar{\alpha} + \bar{\epsilon}_{..}) + k^{-1} \sum_i N^{1/2} \bar{\epsilon}_{i.}^2 \\ & - 2N^{1/2}(\bar{\alpha} + \bar{\epsilon}_{..})k^{-1} \sum_i \bar{\epsilon}_{i.}] \end{aligned}$$

As before, $N^{1/2}(\bar{\alpha} + \bar{\epsilon}_{..}) = o_p(1)$. Also, $k^{-1} \sum_i \bar{\epsilon}_{i.}$ is $o_p(1)$ using the Markov inequality. In the middle term of (3.2.33) let $A'_N = k^{-1} \sum_i N^{1/2} \bar{\epsilon}_{i.}^2$. Since the $\bar{\epsilon}_{i.}$ are independent and identically distributed, assumption (iv)

assures us that $E(A_N^-) = E(N^{1/2} \varepsilon_{1.}^-) = o_p(1)$. Therefore, we see that $A_N^- = o_p(1)$ by again applying the Markov inequality. Combining these results we see that (3.2.33) is equal to $B_2/2 [o_p(1)o_p(1) + o_p(1) - o_p(1)o_p(1)] = o_p(1)$.

Recalling the definition of $\hat{B}_N(Y_i)$ given in (3.2.31), we express the fourth term of (3.2.32) as

$$\begin{aligned}
& k^{-1} N^{1/2} \sum_i ([1 - 2F(\alpha_i + \bar{\varepsilon}_{i.} - \bar{\alpha} - \bar{\varepsilon}_{..})] I(\alpha_i > 0) I(\alpha_i + \bar{\varepsilon}_{i.} - \bar{\alpha} - \bar{\varepsilon}_{..} < 0) \\
& + [2F(\alpha_i + \bar{\varepsilon}_{i.} - \bar{\alpha} - \bar{\varepsilon}_{..}) - 1] I(\alpha_i < 0) I(\alpha_i + \bar{\varepsilon}_{i.} - \bar{\alpha} - \bar{\varepsilon}_{..} > 0)) \\
& \leq 2k^{-1} N^{1/2} \sum_i ([F(\alpha_i) - F(\alpha_i + \bar{\varepsilon}_{i.} - \bar{\alpha} - \bar{\varepsilon}_{..})] I(0 < \alpha_i < \bar{\alpha} + \bar{\varepsilon}_{..} - \bar{\varepsilon}_{i.})) \\
& + [F(\alpha_i + \bar{\varepsilon}_{i.} - \bar{\alpha} - \bar{\varepsilon}_{..}) - F(\alpha_i)] I(\bar{\alpha} + \bar{\varepsilon}_{..} - \bar{\varepsilon}_{i.} < \alpha_i < 0)) \\
& = 2k^{-1} N^{1/2} \sum_i (|(\bar{\varepsilon}_{i.} - \bar{\alpha} - \bar{\varepsilon}_{..}) f(\alpha_i + \tau'_i)| [I(0 < \alpha_i < \bar{\alpha} + \bar{\varepsilon}_{..} - \bar{\varepsilon}_{i.}) \\
& + I(\bar{\alpha} + \bar{\varepsilon}_{..} - \bar{\varepsilon}_{i.} < \alpha_i < 0)]) \\
& \quad \text{where } \tau'_i \text{ is between 0 and } \bar{\varepsilon}_{i.} - \bar{\alpha} - \bar{\varepsilon}_{..}.
\end{aligned}$$

$$\begin{aligned}
(3.2.34) \quad & \leq 2B_1 N^{1/2} |\bar{\alpha} + \bar{\varepsilon}_{..}| k^{-1} \sum_i [I(0 < \alpha_i < \bar{\alpha} + \bar{\varepsilon}_{..} - \bar{\varepsilon}_{i.}) + I(\bar{\alpha} + \bar{\varepsilon}_{..} - \bar{\varepsilon}_{i.} < \alpha_i < 0)] \\
& + 2B_1 N^{1/2} k^{-1} \sum_i |\bar{\varepsilon}_{i.}| [I(0 < \alpha_i < \bar{\alpha} + \bar{\varepsilon}_{..} - \bar{\varepsilon}_{i.}) + I(\bar{\alpha} + \bar{\varepsilon}_{..} - \bar{\varepsilon}_{i.} < \alpha_i < 0)].
\end{aligned}$$

To complete the proof of the lemma we must show both terms in (3.2.34) are $o_p(1)$.

Beginning with the first term, we have seen previously that $N^{1/2} |\bar{\alpha} + \bar{\varepsilon}_{..}| = o_p(1)$. Since the $(\alpha_i, \bar{\varepsilon}_{i.})$ are identically distributed for $i = 1, 2, \dots, k$, we see that

$$\begin{aligned}
& E(k^{-1} \sum_i [I(0 < \alpha_i < \bar{\alpha} + \bar{\varepsilon}_{..} - \bar{\varepsilon}_{i.}) + I(\bar{\alpha} + \bar{\varepsilon}_{..} - \bar{\varepsilon}_{i.} < \alpha_i < 0)]) \\
& = P(0 < \alpha_1 < \bar{\alpha} + \bar{\varepsilon}_{..} - \bar{\varepsilon}_{1.}) + P(\bar{\alpha} + \bar{\varepsilon}_{..} - \bar{\varepsilon}_{1.} < \alpha_1 < 0).
\end{aligned}$$

Using the assumptions of the theorem we have also shown that

$N^{1/2}(\bar{\alpha} + \bar{\varepsilon} - \bar{\varepsilon}_{1.}) = o_p(1)$. This implies that for any $\Delta > 0$, there exists a bounded $D > 0$, such that $P(-D < N^{1/2}(\bar{\alpha} + \bar{\varepsilon} - \bar{\varepsilon}_{1.}) < D) > 1 - \Delta/2$ for large N . Therefore,

$$P(0 < \alpha_1 < \bar{\alpha} + \bar{\varepsilon} - \bar{\varepsilon}_{1.}) \leq P(0 < \alpha_1 < D/N^{1/2}) + P(|N^{1/2}(\bar{\alpha} + \bar{\varepsilon} - \bar{\varepsilon}_{1.})| > D)$$

and thus, for large N ,

$$\begin{aligned} P(0 < \alpha_1 < \bar{\alpha} + \bar{\varepsilon} - \bar{\varepsilon}_{1.}) &\leq P(0 < \alpha_1 < D/N^{1/2}) + \Delta/2 \\ &= G(D/N^{1/2}) - G(0) + \Delta/2. \end{aligned}$$

Since $D/N^{1/2} = o(1)$ and $G(x)$ is continuous, the above quantity can be made arbitrarily small by choosing large enough N . Thus, we have shown that $P(0 < \alpha_1 < \bar{\alpha} + \bar{\varepsilon} - \bar{\varepsilon}_{1.}) = o(1)$. By a similar argument it can be shown that $P(\bar{\alpha} + \bar{\varepsilon} - \bar{\varepsilon}_{1.} < \alpha_1 < 0) = o(1)$. Using the Markov inequality we then see that the first term of (3.2.34) is $2B_1 o_p(1)o_p(1) = o_p(1)$.

For the second term look at

$$\begin{aligned} &E\left(N^{1/2}\sum_i |\bar{\varepsilon}_{1.}| [I(0 < \alpha_1 < \bar{\alpha} + \bar{\varepsilon} - \bar{\varepsilon}_{1.}) + I(\bar{\alpha} + \bar{\varepsilon} - \bar{\varepsilon}_{1.} < \alpha_1 < 0)]\right)^2 \\ &\leq Nk^{-1}E(\bar{\varepsilon}_{1.}^2) + N(k-1)k^{-1}E(|\bar{\varepsilon}_{1.}| [I(0 < \alpha_1 < \bar{\alpha} + \bar{\varepsilon} - \bar{\varepsilon}_{1.}) \\ &\quad + I(\bar{\alpha} + \bar{\varepsilon} - \bar{\varepsilon}_{1.} < \alpha_1 < 0)])|\bar{\varepsilon}_{2.}| [I(0 < \alpha_2 < \bar{\alpha} + \bar{\varepsilon} - \bar{\varepsilon}_{2.}) \\ &\quad + I(\bar{\alpha} + \bar{\varepsilon} - \bar{\varepsilon}_{2.} < \alpha_2 < 0)]]. \end{aligned}$$

Let $[I(0 < \alpha_1 < \bar{\alpha} + \bar{\varepsilon} - \bar{\varepsilon}_{1.}) + I(\bar{\alpha} + \bar{\varepsilon} - \bar{\varepsilon}_{1.} < \alpha_1 < 0)][I(0 < \alpha_2 < \bar{\alpha} + \bar{\varepsilon} - \bar{\varepsilon}_{2.}) + I(\bar{\alpha} + \bar{\varepsilon} - \bar{\varepsilon}_{2.} < \alpha_2 < 0)] = S$ and $(|\bar{\varepsilon}_{1.}| |\bar{\varepsilon}_{2.}|) = R$. Using the Schwartz inequality (Chow and Teicher 1978, Page 104), the expectation of interest is less than or equal to

$$\begin{aligned}
& Nk^{-1} E(\bar{\varepsilon}_{1.}^2) + N(k-1)k^{-1} [E(R^2)]^{1/2} [E(S^2)]^{1/2} \\
& = o(1) + N(k-1)k^{-1} [E(\bar{\varepsilon}_{1.}^2)] [E(S)]^{1/2} \\
& = o(1) + O(1)o(1) = o(1)
\end{aligned}$$

since $E(\bar{\varepsilon}_{1.}^2) = O(N)$ and $E(S) \leq P(0 < \alpha_1 < \bar{\alpha} + \bar{\varepsilon}_{..} - \bar{\varepsilon}_{1.}) + P(\bar{\alpha} + \bar{\varepsilon}_{..} - \bar{\varepsilon}_{1.} < \alpha_1 < 0)$, which is $o(1)$ as shown above. Thus, again applying the Markov inequality, we see that the second term in (3.2.34) is $o_p(1)$ completing the proof of the lemma.

To prove Theorem 3.2.1 we recall that under H_0 : $\theta = 1$,

$$N^{1/2} [T_N(\underline{\varepsilon}, \underline{\alpha}) - 1/4] (48Nk^{-1})^{1/2} \xrightarrow[N \rightarrow \infty]{d} N(0, 1).$$

Theorem 3.2.1 is established by showing

$$(3.2.35) \quad N^{1/2} [T_N(\underline{X}, \underline{Y}) - T_N(\underline{\varepsilon}, \underline{\alpha})] = o_p(1)$$

and applying Slutsky's Theorem.

Using the representations of $T_N(\underline{\varepsilon}, \underline{\alpha})$ and $T_N(\underline{X}, \underline{Y})$ given in (3.2.10) and (3.2.19) respectively, we write the LHS of (3.2.35) as

$$N^{1/2} (A^* + B_{1N}^* + B_{2N}^* + \sum_{h=1}^4 C_{hN}^* - A - B_{1N} - B_{2N} - \sum_{h=1}^4 C_{hN}).$$

In Appendix E it is shown that the C_{hN}^* and C_{hN} terms are all $o_p(N^{-1/2})$. Since $A^* = A$, for large N we can write the LHS of (3.2.35) as

$$N^{1/2} (B_{1N}^* + B_{2N}^* - B_{1N} - B_{2N}) + o_p(1).$$

Using (3.2.22) and (3.2.23) this quantity is equal to

$$(1-\lambda_N)N^{1/2} \left(n^{-1} \sum_j [B(\varepsilon_{1j}) - \bar{B}(\varepsilon_{1j})] - k^{-1} \sum_i [\hat{B}(\alpha_i + \bar{\varepsilon}_{i.}) - \hat{B}(\alpha_i)] \right) + o_p(1).$$

The theorem follows by applying Lemmas 3.2.1 and 3.2.2.

Theorem 3.2.2. Using the model and assumptions described in Section 3.1 and the pseudo-samples given in (3.1.2), if assumptions (i), (ii), and (iii) of Theorem 3.2.1 are satisfied and if

$$(iva) \liminf_{x \rightarrow \infty} \frac{-\ln[1-F(x)]}{2\ln(x)} > 0$$

or

$$(ivb) \int |x|^\psi f(x) dx < \infty \text{ for some } \psi > 0,$$

then

$$N^{1/2} [T_N(\underline{X}', \underline{Y}') - 1/4] (48nk^{-1})^{1/2} \xrightarrow[N \rightarrow \infty]{d} N(0, 1).$$

Indication of Proof: The assumptions of Theorem 3.2.1, where sample means were used to form the pseudo-samples, implied that $\bar{\alpha} = o_p(1)$ and $N^{1/2}\bar{\alpha} = o_p(1)$. Also, for every $i = 1, 2, \dots, k$, that $N^{1/2}\bar{\varepsilon}_{i.} = o_p(1)$, $\bar{\varepsilon}_{i.} = o_p(1)$, and $E[(N^{1/2}\bar{\varepsilon}_{i.})^2]$ is uniformly bounded. These facts were instrumental in the proof of the theorem. In the present theorem, where sample medians are used to form the pseudo-samples, the assumptions produce similar results for $\hat{\alpha}$ and $\hat{\varepsilon}_i$, $i = 1, 2, \dots, k$, (defined in Section 3.1).

Assumption (iii) assures us that $N^{1/2}\hat{\alpha} = o_p(1)$ (see Proposition E.10 in Appendix E) and $N^{1/2}\hat{\varepsilon}_i = o_p(1)$ for every i (Serfling 1980, Page 77). Anderson (1981, Propositions 1 and 2) showed that either of (iva) and (ivb) is a necessary and sufficient condition for $E[(N^{1/2}\hat{\varepsilon}_i)^2]$ to be uniformly bounded for every i (under the assumption that $f(x)$ is

symmetric about zero, Anderson's conditions on α_+ and α_- in his Proposition 1 are equivalent to (iva)). For distributions with finite first moment, (ivb) is obviously satisfied. For the Cauchy distribution it can be shown that (iva) is true and (ivb) is true for $0 < \psi < 1$. Since the sample median is a consistent estimate for the population median and since $F(x)$ represents a distribution symmetric about zero, we see that $\hat{\alpha} = o_p(1)$ and $\hat{\epsilon}_i = o_p(1)$ for $i = 1, 2, \dots, k$.

Therefore, the proof of Theorem 3.2.2 is analogous to the proof of Theorem 3.2.1 with $\bar{\alpha} + \bar{\epsilon}_{..}$ replaced by $\hat{\alpha}$ and $\bar{\epsilon}_1$ replaced by $\hat{\epsilon}_1$ for every i .

The proofs that appear in Appendix E regarding the negligibility of the C^* terms are given utilizing the assumptions of Theorem 3.2.1 and using sample means to form the pseudo-samples. Corresponding proofs using the assumptions of Theorem 3.2.2 and sample medians to form the pseudo-samples are analogous.

Whether to obtain samples like those in (3.1.1) or (3.1.2) will depend for the most part on what is known or believed about the actual distributions of the ϵ_{ij} and α_i . For some distributions, the Cauchy distribution for example, second moments do not exist so samples obtained using medians as in (3.1.2) would be used since the assumptions for Theorem 3.2.1, which pertain to pseudo-samples constructed with sample means, are not met. In those cases where either means or medians could be used, the size of the tails of the distributions would be an important factor in choosing how to obtain the samples. For distributions with heavy tails, when extreme observations are more likely, samples involving medians may be preferred since the median is less affected by extreme observations than the mean. For distributions

with lighter tails, like the normal distribution, the mean may be preferred over the median since in these cases the mean is more efficient (Serfling 1980, Page 86).

Theorems similar to Theorems 3.2.1 and 3.2.2 could possibly be proven if the pseudo-samples involved are obtained by using estimates that have the same type of large sample properties as the sample mean and sample median. Adjustments in the assumptions may have to be made in these cases to ensure that the estimates meet the requirements for proof of a theorem analogous to Theorem 3.2.1 or Theorem 3.2.2.

3.3 Asymptotic Confidence Intervals Using the Modified Ansari-Bradley Statistic

If we could observe the actual values of ϵ_{1j} and α_i as our two samples we could use the procedure developed by Bauer (1972) to construct an exact confidence interval for $\theta = \delta_2/\delta_1$ or any function of θ , such as $\gamma = \theta^2/(\theta^2 + 1) = \delta_2^2/(\delta_1^2 + \delta_2^2)$. Using the values of $W(\epsilon, g)$ for which $\theta = 1$ is not rejected, Bauer derives a confidence interval for θ where the endpoints are particular order statistics of the subset of ratios α_i/ϵ_{1j} which are greater than zero. The choices of the order statistics, and hence the confidence coefficient of the interval, are derived using the tabled distribution of the Ansari-Bradley statistic.

Without the actual ϵ_{1j} and α_i as our two samples we cannot obtain an exact confidence interval for θ . However, we can construct an asymptotic confidence interval using a procedure of Sen (1966) that uses the results of Chernoff and Savage (1958), Hodges and Lehmann (1963), Lehmann (1963), and Sen (1963) among others.

Using Sen's procedure for constructing an asymptotic confidence interval for θ requires a statistic, which we shall call $S_N(\underline{X}, \underline{Y})$, such that $S_N(\theta \underline{X}, \underline{Y})$ is monotonically increasing in θ and, when properly standardized, has an asymptotic normal distribution. These conditions being satisfied, the endpoints of an asymptotic $100(1-\zeta)\%$ confidence interval for θ are

$$(3.3.1) \quad \begin{aligned} \hat{\theta}_{LN} &= \inf \{ \theta : z_N > -z_{\zeta/2} \} \quad \text{and} \\ \hat{\theta}_{UN} &= \sup \{ \theta : z_N < z_{\zeta/2} \}, \end{aligned}$$

where z_N is the standardized version of $S_N(\theta \underline{X}, \underline{Y})$ and $z_{\zeta/2}$ is the $(1-\zeta/2)$ th percentile of the standard normal distribution.

We can apply Sen's procedure using the modified Ansari-Bradley statistic $W(\underline{X}, \underline{Y})$ where \underline{X} and \underline{Y} are samples of observations as in (3.1.1) or (3.1.2). Recalling the description of W in Section 3.1 it is easily seen that $W(\theta \underline{X}, \underline{Y})$ is monotonically increasing in θ . Defining

$$(3.3.2) \quad z_N(\theta \underline{X}, \underline{Y}) = [W(\theta \underline{X}, \underline{Y}) - nN/4](nkN/48)^{-1/2},$$

it follows from Theorem 3.2.1 or Theorem 3.2.2 that $z_N(\theta \underline{X}, \underline{Y})$ has an asymptotic standard normal distribution. Thus, the requirements for the use of Sen's procedure are met and the endpoints of the desired interval are as described in (3.3.1).

In order to derive computational formulas for $\hat{\theta}_{LN}$ and $\hat{\theta}_{UN}$ in our case, we will use a representation of $W(\underline{X}, \underline{Y})$ introduced by Bhattacharyya (1977). Bhattacharyya's representation uses ratios of observation from the two samples in much the same way as in Bauer's (1972) exact confidence interval procedure.

To begin, the X_j and Y_i are adjusted by subtracting the combined sample median. This centers the combined sample around zero but does not change the value of $W(\underline{X}, \underline{Y})$ since all observations are shifted equally. If we let \hat{m} be the combined sample median then we are really dealing with the samples $\underline{X} - \hat{m}\underline{1}$ and $\underline{Y} - \hat{m}\underline{1}$. For ease of exposition we suppress this fact by continuing to refer to the samples as the vectors \underline{X} and \underline{Y} .

We now define some notation as in Bhattacharyya (1977):

$\min W(\underline{X}, \underline{Y})$ = minimum possible value of $W(\underline{X}, \underline{Y})$. Attained when all the X_j have the smallest ranks,

relevant pair: a pair (X_j, Y_i) where X_j and Y_i have the same sign,

(3.3.3) $p(\underline{X}, \underline{Y})$ = number of relevant pairs in the two observed samples,

$p'(\underline{X}, \underline{Y})$ = number of relevant pairs where $X_j/Y_i > 1$,

$p_{\max}(\underline{X}, \underline{Y})$ = maximum possible number of relevant pairs.

Bhattacharyya proved that $W(\underline{X}, \underline{Y})$ can be written in terms of these quantities through the expression

$$(3.3.4) \quad p'(\underline{X}, \underline{Y}) + (1/2)[p_{\max}(\underline{X}, \underline{Y}) - p(\underline{X}, \underline{Y})] + \min W(\underline{X}, \underline{Y}) = W(\underline{X}, \underline{Y}).$$

For ease of exposition we will refer to the quantities defined in (3.3.3) as simply $\min W$, p , p' , and p_{\max} . We also note that if $N = n + k$ is odd, one of the observations in the combined sample will be zero after subtracting \hat{m} . In this case (3.3.4) is obtained by eliminating this observation from consideration in forming the relevant pairs.

The quantities $\min W$ and p_{\max} , defined in (3.3.3), are constants that depend on n and k . If n and k are both even, $\min W = n(n+2)/4$ and $p_{\max}/2 = nk/4$. If n and k are both odd, $\min W = (n+1)^2/4$ and $p_{\max}/2 = (nk-1)/4$. If n is odd and k is even, $\min W = (n-1)(n+1)/4$ and $p_{\max}/2 = k(n-1)/4$ if \hat{m} is an X while $\min W = (n+1)^2/4$ and $p_{\max}/2 = n(k-1)/4$ if \hat{m} is a Y . If n is even and k is odd, $\min W = n^2/4$ and $p_{\max}/2 = k(n-1)/4$ if \hat{m} is an X while $\min W = n(n+2)/4$ and $p_{\max}/2 = n(k-1)/4$ if \hat{m} is a Y . In all of these cases, for large N (which is what we are interested in) $\min W$ behaves like $n^2/4$ and $p_{\max}/2$ behaves like $nk/4$. Thus, for large N , $\min W + p_{\max}/2 \approx nN/4$.

We can now derive the computational formulas for the endpoints (3.3.1) of the interval obtained using Sen's (1966) procedure. First we look at $\hat{\theta}_{LN}$:

$$\begin{aligned}\hat{\theta}_{LN} &= \inf\{\theta: Z_N > -z_{\zeta/2}\} \\ &= \inf\{\theta: Z_N(\theta\underline{X}, \underline{Y}) > -z_{\zeta/2}\} \\ &= \inf\{\theta: [W(\theta\underline{X}, \underline{Y}) - nN/4](nkN/48)^{-1/2} > -z_{\zeta/2}\} \\ &= \inf\{\theta: W(\theta\underline{X}, \underline{Y}) > nN/4 - (z_{\zeta/2})(nkN/48)^{1/2}\},\end{aligned}$$

which, using (3.3.4), can be written as

$$\inf\{\theta: p' + p_{\max}/2 - p/2 + \min W > nN/4 - (z_{\zeta/2})(nkN/48)^{1/2}\}$$

and for large N is equivalent to

$$\inf\{\theta: p' > p/2 - (z_{\zeta/2})(nkN/48)^{1/2}\}.$$

Recalling the definitions of p' and p from (3.3.3) and their dependence on the vectors of observations $\theta\underline{X}$ and \underline{Y} we obtain

$$\begin{aligned}
 \hat{\theta}_{LN} &= \text{Inf}\{\theta: \#(\theta X_j/Y_i > 1) > p/2 - (z_{\zeta/2})(nkN/48)^{1/2}\} \\
 &= \text{Inf}\{\theta: \#(Y_i/X_j < \theta) > p/2 - (z_{\zeta/2})(nkN/48)^{1/2}\} \\
 &= \text{Inf}\{\theta: \text{more than } p/2 - (z_{\zeta/2})(nkN/48)^{1/2} \text{ of the positive } \\
 &\quad (Y_i/X_j) \text{ are less than } \theta\}.
 \end{aligned}$$

If we order the positive (Y_i/X_j) we can apply the above expression to see that

$$\begin{aligned}
 (3.3.5) \quad \hat{\theta}_{LN} &= \text{the } \{p/2 - (z_{\zeta/2})(nkN/48)^{1/2}\}+1 \text{ order statistic} \\
 &\quad \text{of the positive } (Y_i/X_j),
 \end{aligned}$$

where $\{x\} =$ the greatest integer less than or equal to x .

Starting with the definition of $\hat{\theta}_{UN}$ in (3.3.1) and following similar steps we obtain

$$\begin{aligned}
 (3.3.6) \quad \hat{\theta}_{UN} &= \text{the } [p/2 + (z_{\zeta/2})(nkN/48)^{1/2}] + 1 \text{ order statistic} \\
 &\quad \text{of the positive } (Y_i/X_j),
 \end{aligned}$$

where $[x] =$ the greatest integer less than x .

It is a simple matter to convert these endpoints of a confidence interval for θ into endpoints of a confidence interval for $\gamma = \theta^2/(\theta^2+1)$. The endpoints of an asymptotic $100(1-\zeta)\%$ confidence interval for γ are

$$(3.3.7) \quad \hat{\gamma}_{LN} = \hat{\theta}_{LN}^2 / (\hat{\theta}_{LN}^2 + 1) \quad \text{and} \quad \hat{\gamma}_{UN} = \hat{\theta}_{UN}^2 / (\hat{\theta}_{UN}^2 + 1).$$

CHAPTER FOUR MONTE CARLO STUDY

A Monte Carlo study was undertaken to compare the various methods of constructing confidence intervals for the intraclass correlation coefficient, $\rho = \sigma_a^2 / (\sigma_a^2 + \sigma_e^2)$, discussed in this dissertation. Throughout this chapter we will refer to ρ as the parameter of interest even though in some cases we are interested in scale parameters and the parameter of interest is $\gamma = \delta_2^2 / (\delta_1^2 + \delta_2^2)$. We refer only to ρ for ease of presentation and because ρ and γ are numerically equivalent in those cases where both exist.

Using IMSL (International Mathematical and Statistical Libraries) subroutines, random numbers were generated from five distributions which are symmetric about zero. The five distributions used were normal, uniform, logistic, Laplace (double exponential), and Cauchy. In each case the resulting random numbers were used to form responses in the balanced one-way random effects model (without loss of generality we assume $\mu = 0$)

$$z_{ij} = \alpha_i + \varepsilon_{ij} \quad i = 1, 2, \dots, k, \quad j = 1, 2, \dots, n.$$

The nk responses in each model were formed by generating $nk + k$ random numbers from one of the distributions, multiplying nk of these numbers by a constant to obtain the simulated values of the ε_{ij} , multiplying the remaining k numbers by a constant to obtain the simulated values of the α_i , and adding the ε_{ij} and α_i to obtain the simulated responses.

Various multipliers were used to obtain effects with differing values of ρ .

For each of the five distributions, four different size models were generated. The size of the model is determined by k (the number of treatments) and n (the number of observations per treatment). The combinations of k and n used (with k listed first) were (6,12), (12,15), (18,12), and (12,6). For models of size (6,12) and (12,15), numbers were generated for each distribution and multipliers chosen to obtain values for ρ of .10000, .26471, .40000, .50000, .60000, .73529, and .90000. For models of size (18,12) and (12,6), only ρ values of .26471, .50000, and .73529 were used. Since the results for these values of ρ were consistent with the results for the same values of ρ for models of size (6,12) and (12,15), the remaining values of ρ were not used for models (18,12) and (12,6).

For every combination of distribution, model size, and ρ value, 200 sets of responses were generated and confidence intervals for ρ were constructed using each of the methods described in this dissertation. We will use the following conventions when referring to the individual procedures. The Normal procedure refers to that based on normal theory and the F-distribution as discussed in Scheffe' (1959, Pages 221-230). The Arvesen procedure is the procedure based on jackknifed U-statistics presented by Arvesen and Schmitz (1970) which leads to intervals of the form given by (2.2.11). In Section 2.2 we presented two procedures for computing intervals based on U-statistics. The first, which involved a function of U-statistics with an asymptotic normal distribution, produced intervals of the form given by (2.2.10) and will hereafter be referred to as the U-statistic procedure. The second, which we call the

Chi-Square procedure, produces intervals of the form given by (2.2.14) and is based on a function of U-statistics which has an asymptotic χ^2_2 distribution. The procedure presented in Chapter Three based on the Ansari-Bradley statistic using pseudo-samples involving means (as given in (3.1.1)) is called the ABMeans procedure. The corresponding procedure based on pseudo-samples constructed using medians (as given in (3.1.2)) is called the ABMedians procedure.

Intervals based on the Chi-Square procedure were not constructed for the Cauchy distribution. These intervals could not be obtained because of overflow errors encountered during the calculations. Since the Chi-Square procedure is clearly inferior to the others (see discussion below), the omission of these results is inconsequential.

Recall from Chapter Three that confidence intervals constructed using the ABMeans and ABMedians procedures are formed using only the ϵ_{ij} from one treatment. Thus, for each of these procedures there are k possible intervals that could be constructed from the responses in one model. Individually, these k possible intervals, unlike the intervals constructed using the other procedures, do not make full use of all the information contained in the responses. In an attempt to obtain a single interval that does make use of all the information, confidence intervals were also formed in each case using procedures we will call ABMeansC and ABMediansC. These intervals were calculated by averaging the endpoints of the k different intervals that could be formed using the ABMeans and ABMedians procedures respectively. This method of combining the k possible intervals is based on the premise that if only one interval was constructed, using either the ABMeans or ABMedians procedure, it would most likely be constructed using the ϵ_{ij} from a

randomly selected treatment. Thus, any of the k possible treatments would be equally likely to be selected and any of the k possible intervals would be equally valid. Averaging the endpoints of all possible intervals can be thought of as assigning an equal weight to each of the equally likely endpoints.

For each 200 intervals and for each procedure, the empirical confidence coefficient (the number of intervals containing the value of ρ divided by 200), the average length of the 200 intervals, and the standard deviation of the 200 lengths were calculated. These calculations were performed differently for the ABMeans and ABMedians procedures since these procedures produce k possible intervals for each set of responses. Therefore, the empirical confidence coefficient and average lengths reported for these procedures were calculated using $200k$ intervals rather than 200. Also, for each $i = 1, 2, \dots, k$, we calculated the standard deviation of the lengths of the 200 intervals constructed if the ε_{ij} , $j = 1, 2, \dots, n$, were used. The standard deviation reported is the average of these k standard deviations.

A summary of the Monte Carlo study is presented in the following tables. These tables are numbered in such a way that tables including results for a particular distribution or model size can be easily identified. The first position in the number of the table refers to the distribution used to generate the responses according to the following scheme: 1-normal, 2-uniform, 3-logistic, 4-Laplace, and 5-Cauchy. Thus, the higher numbered tables are for distributions with heavier tails (exception is that the normal distribution has heavier tails than the uniform distribution).

Table 1A1
Behavior of nominal 90% confidence intervals for models with
 $k = 6$ treatments, $n = 12$ observations per treatment, and F normal.

		ρ						
		.10000	.26471	.40000	.50000	.60000	.73529	.90000
Arvesen	.91000	.87000	.90000	.88000	.87500	.90000	.94000	
	.5257	.5674	.5995	.6211	.5855	.5551	.3220	
	.312	.226	.209	.194	.195	.217	.187	
Normal	.93000	.90000	.87500	.88500	.88000	.92500	.91000	
	.3845	.5060	.5224	.5149	.4868	.4215	.2178	
	.122	.088	.072	.068	.080	.106	.101	
ABMeans	.82167	.85500	.85750	.83667	.84833	.85833	.89167	
	.4244	.5594	.5987	.6001	.5926	.5669	.3820	
	.217	.174	.168	.158	.166	.193	.237	
ABMedians	.49417	.87167	.93000	.88083	.85250	.79917	.80667	
	.4422	.4992	.5202	.5228	.5228	.5095	.4177	
	.134	.122	.118	.121	.125	.137	.190	
ABMeansC	.87000	.92000	.90500	.92000	.88500	.91500	.95000	
	.4244	.5594	.5987	.6001	.5926	.5669	.3820	
	.153	.107	.114	.112	.133	.165	.216	
ABMediansC	.44000	.91500	.98500	.94500	.93000	.87500	.87000	
	.4422	.4992	.5202	.5228	.5228	.5095	.4177	
	.083	.073	.070	.075	.080	.104	.165	
U-statistic	.72000	.68500	.63500	.62500	.63500	.65000	.69500	
	.2000	.3128	.3376	.3547	.3358	.3021	.1533	
	.110	.111	.114	.110	.113	.112	.089	
Chi-Square	.87500	.81000	.76500	.78000	.70000	.73000	.79000	
	.2625	.4070	.5219	.5422	.3983	.3597	.1820	
	.150	.146	.170	.170	.134	.134	.106	

Note: For each procedure the first row is the empirical confidence coefficient, the second is the average length of the intervals, and the third is the standard deviation of the lengths. Example: For 200 90% confidence intervals constructed using Arvesen's procedure when $\rho = .10000$, the proportion of intervals that contained .10000 was .91000, the average length of the intervals was .5257, and the standard deviation of the lengths was .312.

Table 1A2
Behavior of nominal 95% confidence intervals for models with
 $k = 6$ treatments, $n = 12$ observations per treatment, and F normal.

		ρ						
		.10000	.26471	.40000	.50000	.60000	.73529	.90000
Arvesen	.93000	.91000	.92500	.94000	.94000	.95500	.96000	
	.5871	.6550	.6939	.7207	.6902	.6650	.4275	
	.303	.226	.206	.187	.197	.228	.226	
Normal	.96500	.96500	.93500	.93000	.92500	.95500	.97000	
	.4728	.5980	.6100	.5971	.5636	.4883	.2572	
	.130	.091	.075	.076	.091	.119	.116	
ABMeans	.90667	.92833	.94000	.92917	.92333	.94167	.95500	
	.5487	.6877	.7176	.7218	.7096	.6840	.4978	
	.238	.176	.163	.153	.163	.196	.258	
ABMedians	.68250	.94500	.97250	.94833	.91667	.89833	.91083	
	.5441	.6047	.6247	.6302	.6305	.6165	.5234	
	.143	.128	.119	.121	.125	.143	.209	
ABMeansC	.94500	.96500	.97500	.96500	.97500	.98500	.97500	
	.5487	.6877	.7176	.7218	.7096	.6840	.4978	
	.161	.104	.107	.111	.131	.173	.239	
ABMediansC	.68500	.98000	.99500	.97000	.98000	.97000	.95000	
	.5441	.6047	.6247	.6302	.6305	.6165	.5234	
	.089	.076	.074	.077	.080	.113	.184	
U-statistic	.81500	.73500	.68000	.71000	.74500	.74000	.78500	
	.2307	.3655	.3974	.4190	.5635	.6128	.5737	
	.126	.138	.134	.129	.202	.239	.336	
Chi-Square	.89500	.87500	.81000	.83500	.79500	.80000	.82500	
	.2860	.4478	.5219	.5989	.6271	.6784	.6782	
	.148	.155	.170	.165	.197	.223	.320	

Note: Format of this table is identical to Table 1A1.

Table 1B1
 Behavior of nominal 90% confidence intervals for models with
 $k = 12$ treatments, $n = 15$ observations per treatment, and F normal.

	ρ						
	.10000	.26471	.40000	.50000	.60000	.73529	.90000
Arvesen	.91500	.93000	.96500	.91000	.89000	.92500	.87500
	.2770	.3601	.4142	.3874	.3853	.3110	.1713
	.189	.109	.113	.106	.116	.094	.083
Normal	.91500	.91500	.92500	.90000	.87000	.92500	.86500
	.2190	.3273	.3599	.3564	.3364	.2717	.1394
	.061	.040	.023	.026	.035	.050	.053
ABMeans	.84333	.90667	.91625	.89583	.89750	.92125	.90208
	.3512	.4783	.5244	.5155	.5088	.4498	.2843
	.146	.124	.106	.106	.114	.135	.149
ABMedians	.64458	.85792	.92250	.90417	.90125	.90667	.83375
	.3773	.4479	.4741	.4705	.4683	.4355	.3222
	.117	.099	.085	.086	.088	.104	.136
ABMeansC	.87000	.95000	.98500	.96500	.97000	.98500	.96000
	.3512	.4783	.5244	.5155	.5088	.4498	.2843
	.081	.064	.058	.073	.086	.106	.125
ABMediansC	.61000	.91000	.98500	.96500	.98000	.97500	.89000
	.3773	.4479	.4741	.4705	.4683	.4355	.3222
	.065	.054	.044	.051	.059	.079	.114
U-statistic	.77000	.78500	.82500	.79000	.74000	.82000	.77500
	.1592	.2593	.3070	.2912	.2873	.2305	.1202
	.064	.062	.071	.066	.075	.062	.055
Chi-Square	.88500	.89500	.91000	.89500	.83000	.91000	.82500
	.1956	.3471	.4475	.4434	.4723	.4465	.3454
	.077	.088	.120	.118	.147	.180	.247

Note: Format of this table is identical to Table 1A1.

Table 1B2
Behavior of nominal 95% confidence intervals for models with
 $k = 12$ treatments, $n = 15$ observations per treatment, and F normal.

	ρ						
	.10000	.26471	.40000	.50000	.60000	.73529	.90000
Arvesen	.94500	.95500	.98000	.96000	.95000	.97500	.92500
	.3268	.4317	.4923	.4623	.4603	.3779	.2151
	.193	.123	.126	.117	.130	.111	.104
Normal	.97500	.95000	.95000	.94000	.96000	.95500	.91000
	.2679	.3917	.4255	.4196	.3954	.3194	.1653
	.073	.044	.025	.031	.042	.058	.062
ABMeans	.90708	.94667	.96125	.94708	.95083	.96583	.95000
	.4267	.5610	.6094	.5999	.5928	.5322	.3507
	.163	.129	.109	.111	.119	.146	.170
ABMedians	.75375	.93167	.95875	.94621	.95250	.95750	.90458
	.4473	.5210	.5515	.5451	.5439	.5095	.3892
	.128	.106	.088	.090	.094	.111	.151
ABMeansC	.93500	.97000	1.00000	.98000	1.00000	.99500	.98000
	.4267	.5610	.6094	.5999	.5928	.5322	.3507
	.089	.066	.061	.077	.090	.117	.144
ABMediansC	.78500	.96000	1.00000	.98000	.99000	.99000	.95000
	.4473	.5210	.5515	.5451	.5439	.5095	.3892
	.071	.057	.046	.055	.065	.086	.128
U-statistic	.83500	.86000	.88500	.84500	.80500	.88000	.85500
	.1836	.3076	.3656	.3467	.3423	.2745	.1431
	.074	.074	.084	.078	.089	.074	.066
Chi-Square	.91000	.94500	.92500	.91500	.86500	.93000	.85500
	.2140	.3867	.5061	.5185	.5578	.5552	.4926
	.083	.095	.125	.132	.161	.202	.306

Note: Format for this table is identical to Table 1A1.

Table 1C
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 18$ treatments, $n = 12$ observations per treatment, and F normal.

	90%			95%		
	ρ	ρ	ρ	ρ	ρ	ρ
	.26471	.50000	.73529	.26471	.50000	.73529
Arvesen	.92000	.90500	.92500	.96000	.95000	.94500
	.2936	.3334	.2474	.3512	.3972	.2987
	.095	.089	.058	.110	.102	.068
Normal	.89000	.90000	.88500	.94500	.95000	.95500
	.2695	.2968	.2336	.3207	.3488	.2619
	.031	.014	.036	.035	.016	.043
ABMeans	.88361	.88471	.90555	.93583	.92805	.94805
	.4675	.5085	.4355	.5353	.5781	.5027
	.128	.106	.130	.135	.109	.140
ABMedians	.83444	.89917	.92389	.89472	.94250	.95778
	.4598	.4787	.4156	.5247	.5450	.4792
	.101	.087	.104	.105	.090	.111
ABMeansC	.98000	.97000	.99500	.99000	.98500	1.00000
	.4675	.5085	.4355	.5353	.5781	.5027
	.057	.062	.097	.059	.065	.106
ABMediansC	.92000	.98000	1.00000	.95500	.99000	1.00000
	.4598	.4787	.4156	.5247	.5450	.4792
	.044	.050	.076	.046	.053	.082

Note: For each procedure the first row is the empirical confidence coefficient, the second is the average length of the intervals, and the third is the standard deviation of the lengths. Also for each procedure, the first three columns apply to nominal 90% confidence intervals and the second three columns apply to nominal 95% confidence intervals. Example: For 200 90% confidence intervals constructed using Arvesen's procedure when $\rho = .26471$, the proportion of intervals that contained .26471 was .92000, the average lengths of the intervals was .2936, and the standard deviation of the lengths was .095. For 200 95% confidence intervals constructed under the same conditions, the corresponding values were .96000, .3512, and .110.

Table 1D
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 12$ treatments, $n = 6$ observations per treatment, and F normal.

	90%			95%		
	ρ	ρ	ρ	ρ	ρ	ρ
	.26471	.50000	.73529	.26471	.50000	.73529
Arvesen	.86500	.89000	.93500	.92000	.94000	.97000
	.4310	.4522	.3692	.5054	.5388	.4480
	.164	.121	.125	.169	.135	.145
Normal	.85500	.88000	.90500	.92000	.93500	.96000
	.4043	.4139	.3185	.4786	.4890	.3787
	.065	.041	.068	.076	.047	.079
ABMeans	.82208	.86708	.88125	.90792	.93333	.94792
	.5725	.6026	.5470	.6898	.7212	.6720
	.167	.155	.185	.167	.149	.186
ABMedians	.78000	.88500	.91708	.88958	.94917	.96417
	.5894	.5922	.5387	.7013	.7045	.6555
	.148	.133	.152	.144	.126	.152
ABMeansC	.92000	.97500	1.00000	.97500	1.00000	1.00000
	.5725	.6026	.5470	.6898	.7212	.6720
	.077	.091	.127	.080	.089	.130
ABMediansC	.87000	.98500	.99500	.96000	1.00000	1.00000
	.5894	.5922	.5387	.7013	.7045	.6555
	.061	.073	.099	.062	.070	.102

Note: Format of this table is identical to Table 1C.

Table 2A
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 6$ treatments, $n = 12$ observations per treatment, and F uniform.

		90%		95%	
		ρ	ρ	ρ	ρ
		.26471	.50000	.73529	.26471
Arvesen		.89500	.92500	.92500	.95000
		.5292	.5576	.4680	.6211
		.206	.189	.182	.210
Normal		.96000	.97000	.97000	.99000
		.5195	.5312	.4197	.6124
		.076	.047	.084	.078
ABMeans		.86583	.88083	.87000	.94833
		.5018	.5646	.5194	.6301
		.173	.145	.178	.185
ABMedians		.87583	.90333	.83250	.95083
		.4691	.5075	.4937	.5756
		.122	.115	.127	.125
ABMeansC		.94000	.95000	.95000	.98000
		.5018	.5646	.5194	.6301
		.112	.103	.155	.115
ABMediansC		.93500	.97500	.89000	.97500
		.4691	.5075	.4937	.5756
		.071	.065	.092	.075

Note: Format of this table is identical to Table 1C.

Table 28
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 12$ treatments, $n = 15$ observations per treatment, and F uniform.

	90%			95%		
	ρ	ρ	ρ	ρ	ρ	ρ
	.26471	.50000	.73529	.26471	.50000	.73529
Arvesen	.92000	.94500	.91500	.95000	.96000	.95500
	.3028	.3311	.2369	.3657	.3981	.2885
	.089	.085	.081	.104	.098	.097
Normal	.95500	.95500	.98000	.97500	.98500	1.00000
	.3311	.3633	.2691	.3960	.4274	.3164
	.039	.015	.039	.044	.018	.046
ABMeans	.89792	.90542	.89833	.95208	.94914	.94583
	.4032	.4610	.3715	.4822	.5431	.4458
	.121	.102	.126	.113	.107	.140
ABMedians	.85708	.92708	.90167	.92292	.96833	.95208
	.4194	.4472	.3988	.4924	.5223	.4677
	.096	.085	.113	.102	.087	.119
ABMeansC	.98500	.98000	.98000	.99500	.99000	.99500
	.4032	.4610	.3715	.4822	.5431	.4458
	.063	.071	.111	.068	.074	.124
ABMediansC	.94000	.98500	.96500	.97500	1.00000	.98500
	.4194	.4472	.3988	.4924	.5223	.4677
	.047	.045	.087	.051	.048	.095

Note: Format of this table is identical to Table 1C.

Table 2C
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 18$ treatments, $n = 12$ observations per treatment, and F uniform.

	90%			95%		
	ρ	ρ	ρ	ρ	ρ	ρ
	.26471	.50000	.73529	.26471	.50000	.73529
Arvesen	.93000	.91500	.90000	.97000	.95500	.94500
	.2487	.2543	.1835	.2992	.3054	.2219
	.058	.059	.052	.069	.069	.062
Normal	.96500	.96500	.97000	.98500	.98500	.98500
	.2746	.2994	.2229	.3266	.3516	.2610
	.024	.009	.028	.027	.010	.033
ABMeans	.87922	.88611	.87361	.93139	.92917	.92444
	.4012	.4427	.3527	.4644	.5095	.4133
	.122	.093	.112	.134	.099	.123
ABMedians	.83556	.91639	.90833	.90194	.95417	.94694
	.4311	.4461	.3689	.4938	.5101	.4262
	.097	.083	.102	.101	.086	.107
ABMeansC	.96500	.98000	.98500	.99000	.99500	.99500
	.4012	.4427	.3527	.4644	.5095	.4133
	.048	.058	.094	.052	.063	.105
ABMediansC	.90000	.99000	.98500	.97000	1.00000	.99000
	.4311	.4461	.3689	.4938	.5101	.4262
	.038	.049	.075	.041	.053	.082

Note: Format of this table is identical to Table 1C.

Table 2D
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 12$ treatments, $n = 6$ observations per treatment, and F uniform.

	90%			95%		
	ρ	ρ	ρ	ρ	ρ	ρ
	.26471	.50000	.73529	.26471	.50000	.73529
Arvesen	.91500	.92500	.93500	.93000	.95000	.96500
	.4055	.3816	.2925	.4809	.4580	.3555
	.127	.107	.117	.139	.122	.136
Normal	.93000	.94500	.98000	.97500	.97500	.99500
	.4170	.4201	.3126	.4932	.4201	.3720
	.051	.031	.054	.060	.031	.063
ABMeans	.86458	.84708	.84708	.93583	.92208	.92583
	.5470	.5505	.4681	.6641	.6686	.5916
	.167	.153	.179	.171	.153	.198
ABMedians	.80167	.87000	.89875	.90833	.93917	.95167
	.5858	.5606	.4829	.6975	.6750	.5989
	.149	.132	.159	.143	.128	.166
ABMeansC	.95000	.94500	.97000	.97000	.99000	.99500
	.5470	.5505	.4681	.6641	.6686	.5916
	.076	.101	.140	.079	.102	.160
ABMediansC	.88500	.96500	.99500	.95500	.99500	1.00000
	.5858	.5606	.4829	.6975	.6750	.5989
	.066	.075	.115	.065	.076	.120

Note: Format of this table is identical to Table 1C.

Table 3A
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 6$ treatments, $n = 12$ observations per treatment, and F logistic.

		90%		95%	
		ρ		ρ	
		.26471	.50000	.73529	.26471
Arvesen		.93000	.88000	.91500	.94500
		.5940	.6133	.5258	.6805
		.231	.214	.188	.229
Normal		.88500	.84000	.84500	.97000
		.5039	.5056	.4203	.5959
		.088	.080	.113	.090
ABMeans		.91000	.84833	.88833	.96333
		.6038	.6192	.5601	.7273
		.167	.167	.188	.160
ABMedians		.86167	.90917	.82750	.94083
		.5047	.5248	.5139	.6123
		.120	.116	.140	.124
ABMeansC		.95000	.92500	.94000	.98000
		.6038	.6192	.5601	.7273
		.102	.118	.157	.094
ABMediansC		.91000	.95500	.91000	.99000
		.5047	.5248	.5139	.6123
		.069	.069	.103	.072

Note: Format of this table is identical to Table 1C.

Table 3B
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 12$ treatments, $n = 15$ observations per treatment, and F logistic.

	90%			95%		
	ρ	ρ	ρ	ρ	ρ	ρ
	.26471	.50000	.73529	.26471	.50000	.73529
Arvesen	.92500	.86500	.89000	.96000	.93500	.92500
	.4090	.4386	.3553	.4843	.5177	.4296
	.148	.142	.124	.160	.150	.142
Normal	.86500	.85500	.86500	.91500	.90000	.94500
	.3265	.3528	.2765	.3904	.4160	.3251
	.048	.027	.059	.053	.032	.069
ABMeans	.88333	.90625	.90917	.93875	.95250	.95583
	.5064	.5402	.4661	.5915	.6247	.5495
	.129	.111	.142	.132	.111	.151
ABMedians	.85042	.91333	.89917	.91750	.95625	.95208
	.4645	.4878	.4454	.5407	.5653	.5209
	.099	.086	.111	.103	.086	.117
ABMeansC	.96500	.98000	.97500	.97500	.99500	.98500
	.5064	.5402	.4661	.5915	.6247	.5495
	.072	.067	.110	.074	.069	.120
ABMediansC	.92500	.97000	.97000	.97500	.99500	.98500
	.4645	.4878	.4454	.5407	.5653	.5209
	.051	.048	.086	.053	.050	.092

Note: Format of this table is identical to Table 1C.

Table 3C
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 18$ treatments, $n = 12$ observations per treatment, and F logistic.

	90%			95%		
	ρ	ρ	ρ	ρ	ρ	ρ
	.26471	.50000	.73529	.26471	.50000	.73529
Arvesen	.94000	.88000	.85500	.96000	.93000	.94500
	.3398	.3579	.2846	.4035	.4251	.3437
	.125	.102	.086	.140	.114	.102
Normal	.85000	.84500	.79500	.91500	.93500	.87000
	.2695	.2956	.2240	.3207	.3476	.2624
	.033	.015	.046	.037	.018	.054
ABMeans	.88806	.87694	.89972	.93083	.92805	.94444
	.4874	.5234	.4444	.5559	.5933	.5123
	.130	.108	.140	.135	.110	.148
ABMedians	.84333	.89833	.91305	.90028	.94444	.95472
	.4718	.4842	.4295	.5579	.5508	.4937
	.102	.086	.108	.106	.089	.115
ABMeansC	.98000	.97500	1.00000	1.00000	.98500	1.00000
	.4874	.5234	.4444	.5559	.5933	.5123
	.056	.062	.106	.058	.065	.113
ABMediansC	.94500	.98000	1.00000	.98500	.98500	1.00000
	.4718	.4842	.4295	.5579	.5508	.4937
	.043	.049	.079	.044	.051	.084

Note: Format of this table is identical to Table 1C.

Table 3D
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 12$ treatments, $n = 6$ observations per treatment, and F logistic.

	90%			95%		
	ρ	ρ	ρ	ρ	ρ	ρ
	.26471	.50000	.73529	.26471	.50000	.73529
Arvesen	.89500	.88500	.89500	.92000	.93000	.94000
	.4585	.4588	.3947	.5354	.5444	.4789
	.180	.138	.135	.189	.147	.154
Normal	.88500	.85000	.87500	.95500	.92000	.94000
	.4084	.4149	.3190	.4824	.4904	.3791
	.061	.048	.077	.071	.052	.089
ABMeans	.85208	.86458	.87458	.92750	.93208	.94667
	.5913	.6098	.5440	.7084	.7299	.6679
	.169	.158	.189	.165	.149	.191
ABMedians	.80250	.89125	.89875	.90542	.95292	.96125
	.5962	.5966	.5258	.7099	.7117	.6420
	.150	.136	.161	.143	.128	.162
ABMeansC	.95500	.98000	.99500	.99000	.99000	1.00000
	.5913	.6098	.5440	.7084	.7299	.6679
	.073	.083	.121	.072	.081	.129
ABMediansC	.89000	.99000	.99500	.98500	1.00000	1.00000
	.5962	.5966	.5258	.7099	.7117	.6420
	.064	.063	.099	.063	.065	.101

Note: Format of this table is identical to Table 1C.

Table 4A
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 6$ treatments, $n = 12$ observations per treatment, and F Laplace.

	90%			95%		
	ρ	ρ	ρ	ρ	ρ	ρ
	.26471	.50000	.73529	.26471	.50000	.73529
Arvesen	.90500	.87000	.85500	.93500	.91500	.89500
	.6241	.6529	.5874	.7015	.7484	.6986
	.256	.218	.217	.246	.216	.215
Normal	.85000	.75500	.77500	.96500	.88000	.87500
	.4761	.4938	.4240	.5668	.5766	.4920
	.103	.095	.127	.107	.103	.144
ABMeans	.86333	.85583	.85417	.94083	.93333	.94250
	.6179	.6521	.5711	.7382	.7673	.6890
	.192	.180	.219	.176	.162	.209
ABMedians	.88333	.90833	.79083	.94667	.96833	.89250
	.5146	.5495	.5144	.6193	.6551	.6272
	.128	.122	.152	.134	.119	.153
ABMeansC	.93500	.93000	.93500	.98000	.97000	.98000
	.6179	.6521	.5711	.7382	.7673	.6890
	.123	.131	.177	.111	.118	.170
AEMediansC	.93500	.96500	.91000	.99000	.99000	.95500
	.5146	.5495	.5144	.6193	.6551	.6272
	.077	.074	.108	.081	.076	.109

Note: Format of this table is identical to Table 1C.

Table 4B
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 12$ treatments, $n = 15$ observations per treatment, and F Laplace.

	90%			95%		
	ρ	ρ	ρ	ρ	ρ	ρ
	.26471	.50000	.73529	.26471	.50000	.73529
Arvesen	.88000	.84500	.86000	.93000	.88000	.92000
	.4133	.4696	.4028	.4862	.5520	.4846
	.184	.154	.136	.197	.165	.155
Normal	.78500	.74500	.77500	.86500	.85000	.86500
	.3105	.3473	.2858	.3722	.4102	.3361
	.057	.032	.064	.064	.036	.075
ABMeans	.88333	.89167	.89625	.93792	.94417	.94208
	.5422	.5707	.5126	.6271	.6557	.5965
	.137	.125	.165	.138	.123	.171
ABMedians	.86083	.90458	.87042	.92625	.95042	.92292
	.4775	.5036	.4777	.5532	.5812	.5561
	.108	.094	.120	.112	.099	.126
ABMeansC	.96000	.96500	.98500	.98500	1.00000	.99500
	.5422	.5707	.5126	.6271	.6557	.5965
	.068	.076	.125	.068	.076	.132
ABMediansC	.94500	.98000	.97500	.98000	1.00000	.99000
	.4775	.5036	.4777	.5532	.5812	.5561
	.054	.053	.088	.059	.055	.093

Note: Format of this table is identical to Table 1C.

Table 4C
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 18$ treatments, $n = 12$ observations per treatment, and F Laplace.

	90%			95%		
	ρ	ρ	ρ	ρ	ρ	ρ
	.26471	.50000	.73529	.26471	.50000	.73529
Arvesen	.89500	.91000	.85500	.94500	.94500	.91500
	.3403	.4314	.3330	.4038	.5075	.4008
	.131	.134	.111	.147	.146	.130
Normal	.78500	.76500	.70500	.86500	.82500	.83500
	.2609	.2912	.2245	.3108	.3424	.2630
	.040	.021	.050	.046	.025	.059
ABMeans	.86444	.88583	.88278	.91222	.93305	.92694
	.5268	.5693	.4808	.5974	.6410	.5521
	.135	.118	.155	.138	.118	.162
ABMedians	.83528	.90528	.90194	.89750	.95055	.94250
	.4880	.5164	.4562	.5542	.5837	.5208
	.103	.095	.119	.106	.096	.123
ABMeansC	.93000	.99000	.97500	.97000	1.00000	.99500
	.5268	.5693	.4808	.5974	.6410	.5521
	.055	.060	.107	.057	.061	.115
ABMediansC	.93500	.99000	.99000	.97000	.99000	.99000
	.4880	.5164	.4562	.5542	.5837	.5208
	.041	.052	.078	.043	.054	.082

Note: Format of this table is identical to Table 1C.

Table 4D
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 12$ treatments, $n = 6$ observations per treatment, and F Laplace.

	90%			95%		
	ρ			ρ		
	.26471	.50000	.73529	.26471	.50000	.73529
Arvesen	.88000	.86000	.82500	.92000	.91500	.90000
	.4856	.4955	.4095	.5614	.5847	.4973
	.207	.155	.138	.210	.166	.161
Normal	.80500	.77500	.77000	.88000	.84500	.85000
	.4020	.4066	.3243	.4755	.4806	.3852
	.066	.055	.087	.076	.063	.101
ABMeans	.79583	.82917	.83500	.89000	.91458	.91458
	.6126	.6128	.5404	.7257	.7284	.6617
	.176	.178	.211	.169	.170	.213
ABMedians	.78042	.88042	.87000	.88333	.94833	.93583
	.6115	.6058	.5341	.7218	.7175	.6514
	.142	.146	.178	.133	.137	.179
ABMeansC	.87000	.95000	.97000	.97000	.99000	.99500
	.6126	.6128	.5404	.7257	.7284	.6617
	.084	.096	.137	.082	.097	.140
ABMediansC	.86000	.98500	.99000	.97500	.99500	1.00000
	.6115	.6058	.5341	.7218	.7175	.6514
	.060	.074	.108	.056	.070	.111

Note: Format of this table is identical to Table 1C.

Table 5A
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 6$ treatments, $n = 12$ observations per treatment, and F Cauchy.

		90%			95%		
		ρ			ρ		
		.26471	.50000	.73529	.26471	.50000	.73529
Arvesen		.66500	.56500	.64000	.72500	.62500	.68500
		.5969	.5642	.6561	.6375	.6148	.7219
		.398	.372	.337	.392	.373	.333
Normal		.46500	.30500	.25000	.88500	.42000	.31500
		.3232	.3583	.3679	.4025	.4389	.4432
		.133	.149	.167	.145	.165	.186
ABMeans		.51333	.62083	.61833	.63000	.72500	.72583
		.5537	.5520	.4795	.6663	.6659	.5949
		.330	.326	.344	.327	.322	.351
ABMedians		.56583	.92500	.68167	.71167	.97147	.77333
		.4635	.4952	.5025	.5559	.5944	.6037
		.204	.204	.202	.225	.217	.216
ABMeansC		.47000	.74000	.77000	.58000	.80000	.85000
		.5537	.5520	.4795	.6663	.6659	.5949
		.237	.228	.254	.241	.230	.264
ABMediansC		.57500	.98500	.81000	.76000	1.00000	.90500
		.4635	.4952	.5025	.5559	.5944	.6037
		.117	.109	.108	.132	.116	.116

Note: Format of this table is identical to Table 1C.

Table 5B
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 12$ treatments, $n = 15$ observations per treatment, and F Cauchy.

	90%			95%		
	ρ	ρ	ρ	ρ	ρ	ρ
	.26471	.50000	.73529	.26471	.50000	.73529
Arvesen	.51000	.49000	.46000	.54500	.54500	.49500
	.4659	.4853	.4809	.4993	.5404	.5432
	.415	.389	.362	.426	.405	.382
Normal	.15000	.15000	.11000	.20000	.18500	.15000
	.1617	.1844	.2042	.1986	.2246	.2464
	.089	.107	.112	.103	.124	.130
ABMeans	.50708	.60625	.67542	.62667	.72792	.78792
	.5692	.5391	.4902	.6738	.6581	.6021
	.303	.351	.312	.297	.302	.317
ABMedians	.46375	.90542	.64708	.54667	.95375	.70125
	.3680	.4149	.4213	.4272	.4801	.4895
	.185	.194	.176	.206	.213	.191
ABMeansC	.45000	.74000	.88500	.63000	.84500	.95000
	.5692	.5391	.4902	.6738	.6581	.6021
	.182	.187	.174	.187	.183	.194
ABMediansC	.52000	.97500	.75500	.62500	.99500	.84000
	.3680	.4149	.4213	.4272	.4801	.4895
	.117	.112	.094	.131	.126	.102

Note: Format of this table is identical to Table 1C.

Table 5C
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 18$ treatments, $n = 12$ observations per treatment, and F Cauchy.

	90%			95%		
	ρ	ρ	ρ	ρ	ρ	ρ
	.26471	.50000	.73529	.26471	.50000	.73529
Arvesen	.49000	.45500	.44500	.52000	.50000	.49500
	.4382	.4473	.4502	.4772	.5023	.5340
	.405	.358	.311	.420	.375	.345
Normal	.13000	.11500	.11000	.16500	.13000	.13000
	.1454	.1643	.1746	.1754	.1972	.2084
	.079	.083	.091	.091	.097	.106
ABMeans	.60861	.68250	.69056	.68889	.76000	.75806
	.6172	.6013	.5301	.6817	.6673	.5967
	.284	.293	.314	.277	.282	.310
AEMedians	.47500	.90556	.63417	.55778	.95000	.68056
	.3690	.4034	.3914	.4198	.4571	.4457
	.191	.196	.188	.210	.215	.205
ABMeansC	.66000	.93500	.98000	.79000	.96000	.98500
	.6172	.6013	.5301	.6817	.6673	.5967
	.146	.161	.183	.137	.156	.186
ABMediansC	.57000	.98500	.76500	.64500	1.00000	.79500
	.3690	.4034	.3914	.4198	.4571	.4457
	.148	.151	.141	.165	.167	.157

Note: Format of this table is identical to Table 1C.

Table 5D
Behavior of nominal 90% and 95% confidence intervals for models with
 $k = 12$ treatments, $n = 6$ observations per treatment, and F Cauchy.

		90%			95%		
		ρ			ρ		
		.26471	.50000	.73529	.26471	.50000	.73529
Arvesen		.67500	.53000	.57500	.69000	.56000	.65000
		.5242	.4839	.5166	.5646	.5478	.6164
		.379	.356	.313	.380	.378	.334
Normal		.48000	.23000	.21500	.64500	.30000	.29000
		.3024	.2897	.2953	.3622	.3464	.3518
		.103	.115	.139	.116	.132	.162
ABMeans		.56292	.62333	.69708	.69708	.75458	.80792
		.5701	.5700	.5784	.6920	.6953	.7042
		.309	.326	.344	.304	.317	.310
ABMedians		.58708	.86667	.73042	.69833	.94042	.81417
		.5208	.5092	.5110	.6176	.6024	.6140
		.222	.236	.227	.232	.250	.207
ABMeansC		.62000	.88500	.97500	.80000	.94500	.99500
		.5701	.5700	.5784	.6920	.6953	.7042
		.162	.177	.208	.155	.169	.190
ABMediansC		.61500	.98500	.86500	.81500	1.00000	.92000
		.5208	.5092	.5110	.6176	.6024	.6140
		.163	.182	.153	.179	.202	.169

Note: Format of this table is identical to Table 1C.

The second position in the number of the table is a letter that refers to the size of the model used according to the following scheme: A-(6,12), B-(12,15), C-(18,12), and D-(12,6).

The first four tables (Tables 1A1, 1A2, 1B1, and 1B2) give complete results for all procedures and all ρ values for nominal 90% and 95% confidence intervals constructed using normally distributed responses and model sizes (6,12) and (12,15). The performance of each procedure over the range of ρ values given in these tables held consistently throughout the rest of the study. For this reason, the remaining tables give results only for the ρ values .26471, .50000, and .73529.

First we examine Table 1A1. The Arvesen procedure gives an empirical confidence coefficient that is consistently close to the 90% nominal level over the entire range of ρ values. However, compared to the other procedures, the average length and the standard deviation of the lengths are quite high. The Normal procedure performs well, as it should in this case since the needed assumptions are met, in empirical confidence coefficient, length, and standard deviation. The ABMeans procedure produces results similar to the Arvesen procedure giving slightly lower empirical confidence coefficient, essentially equal length, and less variability of length. The ABMedians procedure gives results similar to the Normal procedure near the center of the range of ρ values except for higher variability of length. The empirical confidence coefficient drops off as ρ gets larger or smaller, especially for $\rho = .10000$.

Due to the method of construction, the ABMeansC and ABMediansC procedures produce intervals with the same average length as the ABMeans and ABMedians procedures. However, the variability of the lengths is

decreased by using the combined procedures. For the ABMeansC procedure the empirical confidence coefficient increases over the ABMeans procedure to the point where it is consistently above the 90% nominal level. However, it is slightly lower at $\rho = .10000$ than it is at other ρ values but then only slightly below the nominal level. The ABMediansC procedure performs very well near the center of the ρ values with empirical confidence coefficient above the nominal level and variability similar to the Normal procedure. However, like the ABMedians procedure, the empirical confidence coefficient drops as ρ moves farther from .50000 and again is especially poor at $\rho = .10000$.

The U-statistic procedure produces short intervals with moderate variability but with empirical confidence coefficient well below the nominal level for all values of ρ . The Chi-Square procedure also produces intervals with consistently low empirical confidence coefficient. The lengths are moderate while the variability of the lengths is quite high.

Table 1A2 contains results for nominal 95% confidence intervals under the same conditions as those used for Table 1A1. These results are consistent with those found in Table 1A1.

Tables 1B1 and 1B2 show what happens if the model size is increased to (12,15). For most procedures empirical confidence coefficients generally get closer to the nominal level than they were for the (6,12) model. For the ABMeansC and ABMediansC procedures, the empirical confidence coefficient rises even higher above the nominal level except for those same ρ values where they were low in the (6,12) model. For all procedures the average lengths and standard deviations of lengths decrease for the larger model though the decrease is not as pronounced

for the AB procedures as it is for the other procedures. Therefore, the ABMeans and ABMedians procedures do not compare as favorably with the Arvesen and Normal procedures as they did when the model was (6,12). The ABMeansC and ABMediansC procedures also have longer lengths than the Arvesen and Normal procedures but with generally higher empirical confidence coefficients.

The U-statistic and Chi-Square procedures also improve as the model size increases but the improvement is not sufficient to raise these procedures to the level of the others. Another aspect to these procedures is the high occurrence of intervals which have endpoints that must be truncated at either 0 or 1 (since $0 < \rho < 1$). This happens more frequently for the Chi-Square procedure than for the U-statistic procedure, sometimes occurring as often as in 60% of the intervals even when $\rho = .50000$. The poor performance of the U-statistic and Chi-Square procedures was consistent over the whole study. For this reason these procedures are not recommended for use and results are not given for them beyond Table 1B2.

Beginning with Table 1C, results for both nominal 90% and 95% confidence intervals are given in the same table for the reduced range of ρ values. Table 1C shows that increasing the model size even further, to (18,12), produces results consistent with the findings when the model size was increased from (6,12) to (12,15). That is, lengths and variability of lengths decrease for all procedures though at a slower rate for the AB procedures. Also, empirical confidence coefficients for the ABMeansC and ABMediansC procedures increase while the empirical confidence coefficients of the other procedures stay near the nominal levels.

The results in Table 1D are also consistent with the results in the previous tables. Table 1D shows that, as would be expected, the performance of the procedures is generally poorer than in Tables 1B1 and 1B2, since n is smaller, but better than in Tables 1A1 and 1A2 since the number of treatments is higher.

Tables 2A through 2D are for effects with uniform distributions. The patterns exhibited in Tables 1A1 through 1D are generally apparent in these tables as well. The most notable exception is that the empirical confidence coefficient for the Normal procedure is noticeably higher than the nominal level. Also, the lengths of the intervals using the Arvesen procedure are essentially the same as in the Normal procedure. However, the Normal procedure is still superior due to the increased empirical confidence coefficient and much less variable lengths.

As the tails of the distributions of the effects get heavier, the empirical confidence coefficient of the Normal procedure decreases. This can be seen in Tables 3A through 3D which deal with effects that have logistic distributions. The results for the other procedures are similar to those seen previously.

For the smaller models, the ABMeansC procedure is apparently superior to the Arvesen procedure. For example, in Table 3A for $\rho = .26471$ and nominal 90% intervals, the ABMeansC procedure produced intervals with higher empirical confidence coefficient, smaller variability of length, and just slightly higher average length than the Arvesen intervals. If the results for nominal 90% intervals for the ABMeansC procedure are compared with nominal 95% intervals for the Arvesen procedure (still $\rho = .26471$) the ABMeansC procedure is better in

all three areas. As the model size increases, the AEMeansC procedure still produces intervals with higher empirical confidence coefficient and less variable lengths than the Arvesen procedure but with clearly higher average lengths (Table 3C).

Tables 4A through 4D give results for effects with Laplace distributions. With even heavier tails the empirical confidence coefficient of the Normal procedure decreases even more. Performance of the other procedures generally follows the patterns discussed earlier.

If the effects have Cauchy distributions then they do not possess finite second moments. Therefore, the only procedures whose assumptions are satisfied are the ABMedians and ABMediansC. This is evident in the results in Tables 5A through 5D where the ABMediansC procedure gives consistently better results than the other procedures. The AEMeans and AEMeansC procedures perform better overall than the Arvesen and Normal procedures and in the larger models (Table 5C for example) are better than The ABMedians and ABMediansC procedures for ρ values away from .50000. For all procedures intervals have more variable lengths when the Cauchy distribution is used.

As we have seen, an overall view of the tables show that it is very difficult to choose a uniformly "best" procedure. Each of the procedures has situations where it performs well and other situations where its performance is poor. With the exception of Cauchy distributed effects (Tables 5A through 5D), the Arvesen procedure produces intervals with confidence coefficient consistently close to the nominal level. However, the lengths of the intervals constructed are quite variable, especially for smaller models.

The Normal procedure produces intervals that are generally narrower and less variable than those produced by the Arvesen procedure but with an inconsistent confidence coefficient. The confidence coefficient ranges from above the nominal level when the uniform distribution is used (Tables 2A through 2D) to well below the nominal level when the Cauchy distribution is used (Tables 5A through 5D).

Another aspect to both the Arvesen and Normal procedures is the possibility of needing to truncate the endpoints of the interval at either 0 or 1. This is necessary more often with the Arvesen procedure than with the Normal procedure and in both cases less than with the Chi-Square procedure. In those cases where truncation was necessary, the length of the interval was calculated using the value of 0 and/or 1.

For the methods using the modified Ansari-Bradley statistics we saw that, due to the method of combining the k possible intervals, the length of the combined interval is the same as the average length of all k possible intervals. Therefore, the average lengths reported in the tables are identical for the ABMeans and ABMeansC procedures as well as for the ABMedians and ABMediansC procedures. Since the tables also showed that combining the intervals produces less variable lengths and consistently higher confidence coefficient, it is recommended that the ABMeansC and ABMediansC procedures be used rather than the ABMeans or ABMedians procedures.

The ABMeansC procedure produces intervals with confidence coefficient consistently higher than the nominal level, even for the smaller models, except when $\rho = .10000$. Even when $\rho = .10000$ the confidence coefficient is only slightly below the nominal level. As with the Arvesen and Normal procedures, the performance of the ABMeansC

procedure is poorer when the effects have Cauchy distributions although the drop-off in performance is less severe. The average lengths of the intervals from the ABMeansC procedure are approximately the same as the average lengths from the Arvesen procedure for small models but do not decrease as quickly as the model size increases.

The ABMediansC procedure produces intervals with generally shorter and less variable lengths than the ABMeansC procedure. The confidence coefficient is consistently above the nominal level for ρ values near .50000 but falls as ρ moves toward 0 or 1. The drop is quite severe as ρ approaches .10000.

As with the ABMeansC procedure, the reduction in average length and standard deviation of length as the model size increases is not as rapid for the ABMediansC procedure as it is with the Arvesen and Normal procedures. However, unlike the Arvesen and Normal procedures, the ABMeansC or ABMediansC procedures will always produce intervals with endpoints between 0 and 1 due to their methods of construction (see Section 3.3).

The choice of which procedure to use to construct a confidence interval for ρ will really depend on how much is known or is being assumed about the model. If it is assumed that the effects have a distribution similar to a uniform or normal distribution, then the Normal procedure is clearly superior (Tables 1A1 through 2D) since it produces narrow intervals with confidence coefficient close to or greater than the nominal level. However, the Normal procedure is not recommended if the effects might have distributions with heavy tails.

If it is believed that ρ is near .50000 and nothing is known about the distribution of the effects, then the ABMediansC procedure is

recommended since, for every distribution including Cauchy, the method performs very well for values of ρ near .50000. However, this method is not recommended if ρ is thought to be near .10000.

If nothing is known about the distribution of the effects or the value of ρ , but moments are assumed to exist, then the Arvesen procedure or the ABMeansC procedure should be used. These procedures gave the most consistent performance over the whole range of situations. For smaller models the ABMeansC procedure would be recommended since it provides less variable intervals with higher confidence coefficient than the Arvesen procedure with little or no increase in length. For larger models the disparity in length and confidence coefficient between the two procedures increases. If a high confidence coefficient is desired, then the ABMeansC procedure should be used. If a shorter length is desired, then the Arvesen procedure will produce such an interval but with more variation in the lengths and a smaller confidence coefficient.

If it is believed that the effects may have a very heavy tailed distribution, such as Cauchy, then either the ABMeansC or ABMediansC procedures should be used since their performance is superior to the other procedures in this case.

The overall performance of the ABMeansC and ABMediansC procedures is such that they merit serious consideration when a confidence interval for ρ is desired. For distributions of all types and for all but extreme values of ρ , these procedures produce intervals that compare favorably with intervals produced by other procedures and, in many cases, are superior. This is especially true when the model size is small. This conclusion is apparently valid even when the assumptions

necessary to apply the other procedures are met. For example, compare the performance of the AEMediansC and Normal procedures when the effects have normal distributions and the model size is (6,12) (Tables 1A1 and 1A2). Yet the AB procedures can sometimes be validly implemented under less restrictive assumptions than the competing techniques.

As we have seen, one of the points to consider when choosing a procedure to use is the assumptions necessary for valid implementation of the procedure. These assumptions are reviewed in the following chapter.

CHAPTER FIVE SUMMARY

In this dissertation we have derived and studied various methods of measuring the proportion of the total variability in the responses from a balanced one-way random effects model,

$$z_{ij} = \mu + \alpha_i + \epsilon_{ij} \quad i = 1, 2, \dots, k, \quad j = 1, 2, \dots, n,$$

that is attributable to the treatments. These methods require different assumptions and therefore, theoretically, can only be used if the appropriate assumptions are met.

The ABMeans and ABMedians procedures (and thus also the ABMeansC and ABMediansC procedures) derived in Chapter Three require the ϵ_{ij} and α_i to possess continuous distributions that are symmetric about zero and that differ only by a scale parameter. Both procedures also require the distributions to have bounded, continuous densities that are positive at zero with bounded, continuous first derivatives. The ABMeans procedure requires the distributions to have finite second moments while the ABMedians procedure requires either $\int |x|^\psi f(x)dx < \infty$ for some $\psi > 0$ or $\liminf_{x \rightarrow \infty} -\ln[1-F(x)][2\ln(x)]^{-1} > 0$. Both procedures are asymptotic as both k (number of treatments) and n (number of observations per treatment) go to infinity. However, as we saw in Chapter Four, Monte Carlo studies show that the procedures perform quite well for small values of n and k .

These assumptions are a broadening of the assumptions used in the classical, normal theory analysis of the balanced one-way random effects model. In the classical analysis the effects are assumed to have normal distributions with zero means and finite variances. The assumptions for the ABMeans and ABMedians procedures allow the effects to have other symmetric distributions and, in the ABMedians procedure, does not require finite second moments.

The U-statistic and Chi-Square procedures derived in Chapter Two, as well as the Arvesen procedure described in Chapter Two, also require the ϵ_{ij} and α_i to have continuous distributions. These distributions must have mean zero and finite fourth moments but need not be symmetric nor of the same family. These procedures only require k to go to infinity rather than both k and n . In some senses this is a more reasonable approach since increasing k is sufficient to obtain more information about both the treatment and error effects. Therefore, the procedures involving U-statistics could be used in some situations where the ABMeans and ABMedians procedures could not.

Of the procedures derived in this dissertation the ABMeansC and ABMediansC procedures produced the most promising results. Future research could include trying to find other ways of combining the individual intervals that would produce a narrower interval, even if the empirical confidence coefficient is decreased. The ABMeansC and ABMediansC procedures produce intervals that, for the most part, have empirical confidence coefficient far above the nominal level. It would be desirable to obtain intervals, presumably shorter, with confidence coefficients nearer to the nominal levels.

Other possible areas of future research would be to extend the procedures to the unbalanced model and to two-way and more complex models. Formation of pseudo-samples using quantities other than sample means or sample medians could also be examined.

APPENDIX A
VARIANCES AND COVARIANCE OF U_1 AND U_2

Using the balanced one-way random effects model under the assumptions given in Section 2.2, we recall that U_1 , given in (2.2.1), is a U-statistic based on a kernel of degree $s = 1$. Therefore, from Result 2.1.2

$$(A.1) \quad \lim_{k \rightarrow \infty} k \text{Var}(U_1) \equiv \xi_{11},$$

where, using (2.1.2),

$$(A.2) \quad \xi_{11} = E[h_1^2(z_1)] - (2\sigma_\epsilon^2)^2.$$

Recalling from the assumptions in Section 2.2 that the ϵ_{ij} and α_i are mutually independent with mean zero we obtain

$$\begin{aligned} E[h_1^2(z_1)] &= \binom{n}{2}^{-2} E[\sum_{j < j'} (z_{1j} - z_{1j'})^2]^2 \\ &= 4[n^2(n-1)^2]^{-1} [\binom{n}{2}(\binom{2}{0})(\binom{n-2}{0}) E(z_{11} - z_{12})^4 \\ &\quad + \binom{n}{2}(\binom{2}{1})(\binom{n-2}{1}) E[(z_{11} - z_{12})^2 (z_{11} - z_{13})^2] \\ &\quad + \binom{n}{2}(\binom{2}{0})(\binom{n-2}{2}) E[(z_{11} - z_{12})^2 (z_{13} - z_{14})^2]] \\ &= 2[n(n-1)]^{-1} [E(\epsilon_{11} - \epsilon_{12})^4 \\ &\quad + 2(n-2)E[(\epsilon_{11} - \epsilon_{12})^2 (\epsilon_{11} - \epsilon_{13})^2] \\ &\quad + (1/2)(n-2)(n-3)E[(\epsilon_{11} - \epsilon_{12})^2 (\epsilon_{13} - \epsilon_{14})^2]] \end{aligned}$$

$$\begin{aligned}
&= 2[n(n-1)]^{-1} [(2\phi_4 + 6\sigma_\varepsilon^4) + 2(n-2)(\phi_4 + 3\sigma_\varepsilon^4) \\
&\quad + (1/2)(n-2)(n-3)(4\sigma_\varepsilon^4)] \\
&= 4\phi_4/n + [n(n-1)]^{-1}(4n^2 - 8n + 12)\sigma_\varepsilon^4
\end{aligned}$$

and therefore, referring to Result 2.1.2, (A.1), and (A.2),

$$\begin{aligned}
\lim_{k \rightarrow \infty} k \text{Var}(U_1) &= 4\phi_4/n + [n(n-1)]^{-1}(4n^2 - 8n + 12)\sigma_\varepsilon^4 - 4\sigma_\varepsilon^4 \\
&= 4n^{-1}[\phi_4 + \sigma_\varepsilon^4(3-n)(n-1)^{-1}],
\end{aligned}$$

establishing (2.2.5).

Now recall that U_2 , given (2.2.2), is a U-statistic of degree $s = 2$. We again use Result 2.1.2 to get

$$(A.3) \quad \lim_{k \rightarrow \infty} k \text{Var}(U_2) \equiv 4\xi_{12},$$

where in this case, again using (2.1.2),

$$(A.4) \quad \xi_{12} = E[h_2(z_1, z_2)h_2(z_1, z_3)] - (2\sigma_\alpha^2 + 2\sigma_\varepsilon^2)^2.$$

Calculating the expectation on the RHS of (A.4) gives

$$\begin{aligned}
&E[h_2(z_1, z_2)h_2(z_1, z_3)] \\
&= n^{-4}E[(\sum_{jj'}(z_{1j}-z_{2j'})^2)(\sum_{jj'}(z_{1j}-z_{3j'})^2)] \\
&= n^{-4}(n^3E[(z_{11}-z_{21})^2(z_{11}-z_{31})^2] \\
&\quad + n^3(n-1)E[(z_{11}-z_{21})^2(z_{12}-z_{31})^2]) \\
&= n^{-1}(E[(\alpha_1-\alpha_2+\varepsilon_{11}-\varepsilon_{21})^2(\alpha_1-\alpha_3+\varepsilon_{11}-\varepsilon_{31})^2] \\
&\quad + (n-1)E[(\alpha_1-\alpha_2+\varepsilon_{11}-\varepsilon_{21})^2(\alpha_1-\alpha_3+\varepsilon_{12}-\varepsilon_{31})^2])
\end{aligned}$$

$$\begin{aligned}
&= n^{-1} [(\eta_4 + \phi_4 + 3\sigma_\alpha^4 + 3\sigma_\varepsilon^4 + 12\sigma_\alpha^2\sigma_\varepsilon^2) \\
&\quad + (n-1)(\eta_4 + 3\sigma_\alpha^4 + 4\sigma_\varepsilon^4 + 8\sigma_\alpha^2\sigma_\varepsilon^2)]
\end{aligned}$$

and therefore, referring to Result 2.1.2, (A.3), and (A.4),

$$\lim_{k \rightarrow \infty} k \text{Var}(U_2) = 4(\eta_4 + \phi_4/n - \sigma_\alpha^4 - \sigma_\varepsilon^4/n + 4\sigma_\alpha^2\sigma_\varepsilon^2/n),$$

establishing (2.2.6).

Finally, observe from Result 2.1.4 that

$$(A.5) \quad \lim_{k \rightarrow \infty} k \text{Cov}(U_1, U_2) = 2\xi_1^{(1,2)},$$

where, using (2.1.3),

$$(A.6) \quad \xi_1^{(1,2)} = E[h_1(z_1)h_2(z_1, z_2)] - (2\sigma_\varepsilon^2)(2\sigma_\alpha^2 + 2\sigma_\varepsilon^2).$$

Again, first looking at the expectation on the RHS of (A.6), we obtain

$$\begin{aligned}
&E[h_1(z_1)h_2(z_1, z_2)] \\
&= 2n^{-3}(n-1)^{-1}E[\sum_{j < j'}(z_{1j} - z_{1j'})^2 \sum_{jj'}(z_{1j} - z_{2j'})^2] \\
&= 2n^{-3}(n-1)^{-1}[2n\binom{n}{2}E[(z_{11} - z_{12})^2(z_{11} - z_{21})^2] \\
&\quad + n\binom{n}{2}\binom{n-2}{1}E[(z_{11} - z_{12})^2(z_{13} - z_{21})^2]] \\
&= n^{-1}[2E[(\varepsilon_{11} - \varepsilon_{12})^2(\alpha_1 - \alpha_2 + \varepsilon_{11} - \varepsilon_{21})^2] \\
&\quad + (n-2)E[(\varepsilon_{11} - \varepsilon_{12})^2(\alpha_1 - \alpha_2 + \varepsilon_{13} - \varepsilon_{21})^2]] \\
&= n^{-1}[2(\phi_4 + 3\sigma_\varepsilon^4 + 4\sigma_\alpha^2\sigma_\varepsilon^2) + (n-2)(4\sigma_\varepsilon^4 + 4\sigma_\alpha^2\sigma_\varepsilon^2)]
\end{aligned}$$

and therefore, referring to Result 2.1.4, (A.5) and (A.6),

$$\lim_{k \rightarrow \infty} k \text{Cov}(U_1, U_2) = 4n^{-1}(\phi_4 - \sigma_\varepsilon^4),$$

establishing (2.2.7).

APPENDIX B
THE RELATIONSHIP BETWEEN U_1 , U_2 , MST, AND MSE

This appendix establishes the relationship between U_1 and U_2 , given in (2.2.1) and (2.2.2), and MST and MSE, the mean squares from the usual one-way analysis of variance table (Scheffé' 1959, Page 225).

First, we expand U_1 , U_2 , MST, and MSE so that each is written completely in terms of the quantities α_i and ϵ_{ij} . While this is not necessary in order to see the relationship between U_1 and MSE, it facilitates establishment of the overall relationship between the statistics.

From (2.2.1) we see that

$$\begin{aligned}
 U_1 &= 2[nk(n-1)]^{-1} \sum_i \sum_{j < j'} (\epsilon_{ij} - \epsilon_{ij'})^2 \\
 (B.1) \quad &= 2[nk(n-1)]^{-1} \sum_i \sum_{j < j'} [(\epsilon_{ij}^2 + \epsilon_{ij'}^2) - 2\epsilon_{ij}\epsilon_{ij'}] \\
 &= 2(nk)^{-1} \sum_i \epsilon_{ij}^2 - 2[nk(n-1)]^{-1} \sum_i \sum_{j \neq j'} \epsilon_{ij}\epsilon_{ij'}.
 \end{aligned}$$

Letting $\bar{Z}_{i.} = n^{-1} \sum_j Z_{ij}$ and $\bar{\epsilon}_{i.} = n^{-1} \sum_j \epsilon_{ij}$ it follows that

$$MSE = [k(n-1)]^{-1} \sum_{ij} (Z_{ij} - \bar{Z}_{i.})^2. \quad \text{Expanding, we obtain}$$

$$\begin{aligned}
 MSE &= [k(n-1)]^{-1} \sum_{ij} (\epsilon_{ij} - \bar{\epsilon}_{i.})^2 \\
 &= [k(n-1)]^{-1} \sum_{ij} (\epsilon_{ij}^2 + \bar{\epsilon}_{i.}^2 - 2\epsilon_{ij}\bar{\epsilon}_{i.})
 \end{aligned}$$

$$\begin{aligned}
 (B.2) \quad &= [k(n-1)]^{-1} [\sum_{ij} \epsilon_{ij}^2 + n^{-1} \sum_i (\sum_j \epsilon_{ij})^2 - 2n^{-1} \sum_i \sum_j \epsilon_{ij} \sum_{j'} \epsilon_{ij'}] \\
 &= [k(n-1)]^{-1} [\sum_{ij} \epsilon_{ij}^2 - n^{-1} \sum_i (\sum_j \epsilon_{ij})^2] \\
 &= [k(n-1)]^{-1} [(\sum_{ij} \epsilon_{ij}^2 - n^{-1} \sum_{ij} \epsilon_{ij}^2) - n^{-1} \sum_i \sum_{j \neq j'} \epsilon_{ij} \epsilon_{ij'}] \\
 &= (nk)^{-1} \sum_{ij} \epsilon_{ij}^2 - [nk(n-1)]^{-1} \sum_i \sum_{j \neq j'} \epsilon_{ij} \epsilon_{ij'}.
 \end{aligned}$$

Thus, from (B.1) and (B.2) it follows that

$$\text{MSE} = U_1/2.$$

In the same manner, from (2.2.2) we see that

$$\begin{aligned}
 U_2 &= 2[n^2 k(k-1)]^{-1} \sum_{i < i'} \sum_{j} \sum_{j'} (\alpha_i - \alpha_{i'} + \epsilon_{ij} - \epsilon_{i'j'})^2 \\
 &= 2[n^2 k(k-1)]^{-1} \sum_{i < i'} \sum_{j} \sum_{j'} [(\alpha_i^2 + \alpha_{i'}^2) \\
 &\quad + (\epsilon_{ij}^2 + \epsilon_{i'j'}^2) - 2(\alpha_i \alpha_{i'}) + 2(\alpha_i \epsilon_{ij} + \alpha_{i'} \epsilon_{i'j'}) \\
 &\quad - 2(\alpha_i \epsilon_{i'j'} + \alpha_{i'} \epsilon_{ij}) - 2(\epsilon_{ij} \epsilon_{i'j'})] \\
 (B.3) \quad &= 2k^{-1} \sum_i \alpha_i^2 + 2(nk)^{-1} \sum_{ij} \epsilon_{ij}^2 \\
 &\quad - 4[k(k-1)]^{-1} \sum_{i < i'} \alpha_i \alpha_{i'} + 4(nk)^{-1} \sum_{ij} \alpha_i \epsilon_{ij} \\
 &\quad - 4[nk(k-1)]^{-1} \sum_{i \neq i'} \sum_{j} \alpha_i \epsilon_{i'j} \\
 &\quad - 4[n^2 k(k-1)]^{-1} \sum_{i < i'} \sum_{j} \sum_{j'} \epsilon_{ij} \epsilon_{i'j'}.
 \end{aligned}$$

Letting $\bar{Z}_{..} = (nk)^{-1} \sum_{ij} Z_{ij}$, $\bar{\epsilon}_{..} = (nk)^{-1} \sum_{ij} \epsilon_{ij}$, and $\bar{\alpha} = k^{-1} \sum_i \alpha_i$, it follows that $\text{MST} = n(k-1)^{-1} \sum_i (\bar{Z}_{i..} - \bar{Z}_{..})^2$. Expanding, we obtain

$$\begin{aligned}
MST &= n(k-1)^{-1} \sum_i (\alpha_i - \bar{\alpha} + \bar{\varepsilon}_{i..} - \bar{\varepsilon}_{..})^2 \\
&= n(k-1)^{-1} \sum_i [\alpha_i^2 + \bar{\alpha}^2 + \bar{\varepsilon}_{i..}^2 + (\bar{\varepsilon}_{..}^2 - 2\bar{\varepsilon}_{i..}\bar{\varepsilon}_{..}) - 2\bar{\alpha}\alpha_i \\
&\quad + 2\alpha_i\bar{\varepsilon}_{i..} - (2\alpha_i\bar{\varepsilon}_{..} + 2\bar{\alpha}\bar{\varepsilon}_{i..} - 2\bar{\alpha}\bar{\varepsilon}_{..})] \\
&= n(k-1)^{-1} [\sum_i \alpha_i^2 + k^{-1}(\sum_i \alpha_i)^2 + n^{-2} \sum_{i,j} (\sum_j \varepsilon_{ij})^2 \\
&\quad - (n^2k)^{-1} (\sum_{i,j} \varepsilon_{ij})^2 - 2k^{-1} (\sum_i \alpha_i)^2 \\
&\quad + 2n^{-1} \sum_i \sum_j \varepsilon_{ij} - 2(nk)^{-1} (\sum_i \alpha_i) (\sum_{i,j} \varepsilon_{ij})] \\
&= n(k-1)^{-1} [[1+k^{-1}-2k^{-1}] \sum_i \alpha_i^2 \\
&\quad + [n^{-2} - (n^2k)^{-1}] \sum_{i,j} \varepsilon_{ij}^2 + [k^{-1} - 2k^{-1}] \sum_{i \neq i'} \alpha_i \alpha_{i'} \\
&\quad + [2n^{-1} - 2(nk)^{-1}] \sum_{i,j} \alpha_i \varepsilon_{ij} + [-2(nk)^{-1}] \sum_{i \neq i'} \sum_j \alpha_i \varepsilon_{i'j} \\
&\quad + [-(n^2k)^{-1}] \sum_{i \neq i'} \sum_{j,j'} \varepsilon_{ij} \varepsilon_{i'j'} \\
&\quad + [n^{-2} - (n^2k)^{-1}] \sum_{i,j \neq j'} \varepsilon_{ij} \varepsilon_{ij'}] \\
&= nk^{-1} \sum_i \alpha_i^2 + (nk)^{-1} \sum_{i,j} \varepsilon_{ij}^2 - 2n[k(k-1)]^{-1} \sum_{i < i'} \alpha_i \alpha_{i'} \\
&\quad + 2k^{-1} \sum_{i,j} \alpha_i \varepsilon_{ij} - 2[k(k-1)]^{-1} \sum_{i \neq i'} \sum_j \alpha_i \varepsilon_{i'j} \\
&\quad - 2[nk(k-1)]^{-1} \sum_{i < i'} \sum_{j,j'} \varepsilon_{ij} \varepsilon_{i'j'} \\
&\quad + (nk)^{-1} \sum_{i,j \neq j'} \varepsilon_{ij} \varepsilon_{ij'}.
\end{aligned} \tag{B.4}$$

It now follows from (B.1), (B.3), and (B.4) that

$$\begin{aligned} \text{MST} - nU_2/2 &= (1-n)(nk)^{-1} \sum_{ij} \varepsilon_{ij}^2 + (nk)^{-1} \sum_i \sum_{j \neq j'} \varepsilon_{ij} \varepsilon_{ij'} \\ &= (1-n)U_1/2 \end{aligned}$$

and thus

$$\text{MST} = [nU_2 + (1-n)U_1]/2.$$

APPENDIX C
A CONSISTENT ESTIMATE FOR σ_T

The confidence interval for ρ given in (2.2.10) involves an estimate for the asymptotic standard deviation of U_1/U_2 . The form of that estimate is derived in this Appendix using the model and assumptions from Section 2.2.

From (2.2.8) and (2.2.9) it follows that

$$(C.1) \quad \sigma_T^2 = \underline{a}' \underline{\Lambda} \underline{a} = \sigma_{11} (2\sigma_\alpha^2 + 2\sigma_\varepsilon^2)^{-2} + 4\sigma_{22} \sigma_\varepsilon^4 (2\sigma_\alpha^2 + 2\sigma_\varepsilon^2)^{-4} - 4\sigma_{12} \sigma_\varepsilon^2 (2\sigma_\alpha^2 + 2\sigma_\varepsilon^2)^{-3}.$$

Theorem 2.1.3 gives conditions under which U-statistics converge almost surely to their expectation. Using (2.2.3) and (2.2.4), it follows that for any number c ,

$$(C.2) \quad U_2^c \xrightarrow{k \rightarrow \infty} (2\sigma_\alpha^2 + 2\sigma_\varepsilon^2)^c$$

and

$$(C.3) \quad U_1^c \xrightarrow{k \rightarrow \infty} (2\sigma_\varepsilon^2)^c.$$

If consistent estimates of σ_{11} , σ_{22} , and σ_{12} can be found, they can be combined with U_1 and U_2 to form a consistent estimate of σ_T^2 as given in (C.1).

We now turn our attention to finding consistent estimates for σ_{11} , σ_{22} , and σ_{12} . Note that from (2.2.5), (A.1), and (A.2), $\sigma_{11} = E[h_1^2(z_1)] - (2\sigma_\varepsilon^2)^2$. Defining

$$(C.4) \quad \hat{\sigma}_{11} = k^{-1} \sum_i [h_1(\underline{Z}_i) - U_1]^2,$$

we obtain the following result.

Result C.1. Defining $\hat{\sigma}_{11}$ as in (C.4), $\hat{\sigma}_{11} \xrightarrow[k \rightarrow \infty]{a.s.} \sigma_{11}$.

Proof: Expanding the RHS of (C.4) we obtain

$$\begin{aligned} \hat{\sigma}_{11} &= k^{-1} \sum_i h_1^2(\underline{Z}_i) - 2U_1 k^{-1} \sum_i h_1(\underline{Z}_i) + U_1^2 \\ &= k^{-1} \sum_i h_1^2(\underline{Z}_i) - U_1^2 \\ &\xrightarrow[k \rightarrow \infty]{a.s.} E[h_1^2(\underline{Z}_1)] - (2\sigma_\epsilon^2)^2. \end{aligned}$$

The last step is justified by using (C.3) with $c = 2$ and noting that $k^{-1} \sum_i h_1^2(\underline{Z}_i)$ is a U-statistic and applying Theorem 2.1.3.

Now note that from (2.2.7), (A.5), and (A.6),

$$\sigma_{12} = 2(E[h_1(\underline{Z}_1)h_2(\underline{Z}_1, \underline{Z}_2)] - (2\sigma_\epsilon^2)(2\sigma_\alpha^2 + 2\sigma_\epsilon^2)). \text{ Defining}$$

$$(C.5) \quad \hat{\sigma}_{12} = 2([k(k-1)]^{-1} \sum_{i \neq i'} \sum h_1(\underline{Z}_i)h_2(\underline{Z}_{i'}, \underline{Z}_{i'}) - U_1 U_2),$$

we obtain the following result.

Result C.2. Defining $\hat{\sigma}_{12}$ as in (C.5), $\hat{\sigma}_{12} \xrightarrow[k \rightarrow \infty]{a.s.} \sigma_{12}$.

Proof: We rewrite $\hat{\sigma}_{12}$ as

$$\begin{aligned} \hat{\sigma}_{12}/2 &= 2[k(k-1)]^{-1} \sum_{i < i'} (1/2)[h_1(\underline{Z}_i)h_2(\underline{Z}_{i'}, \underline{Z}_{i'}) \\ &\quad + h_1(\underline{Z}_{i'})h_2(\underline{Z}_i, \underline{Z}_{i'})] - U_1 U_2. \end{aligned}$$

The first term on the RHS is a U-statistic and therefore, by Theorem 2.1.3, converges almost surely to the expectation of its kernel which is

$$(1/2)E[h_1(\underline{z}_1)h_2(\underline{z}_1, \underline{z}_2) + h_1(\underline{z}_2)h_2(\underline{z}_2, \underline{z}_1)] \\ = E[h_1(\underline{z}_1)h_2(\underline{z}_1, \underline{z}_2)].$$

Application of (C.2) and (C.3) with $c = 1$ completes the proof.

Finally, note that (2.2.6), (A.3), and (A.4) imply that

$$\sigma_{22} = 4(E[h_2(\underline{z}_1, \underline{z}_2)h_2(\underline{z}_1, \underline{z}_3)]) - (2\sigma_a^2 + 2\sigma_e^2)^2. \quad \text{Defining}$$

$$(C.6) \quad g(\underline{z}_i) = (k-1)^{-1} \sum_{i' \neq i} h_2(\underline{z}_i, \underline{z}_{i'})$$

and

$$(C.7) \quad \delta_{22} = 4k^{-1} \sum_i [g(\underline{z}_i) - u_2]^2,$$

we obtain the following result.

Result C.3. Defining $g(\underline{z}_i)$ as in (C.6) and δ_{22} as in (C.7),

$$\delta_{22} \xrightarrow[k \rightarrow \infty]{a.s.} \sigma_{22}.$$

Proof: Expanding (C.7) we obtain

$$\begin{aligned} \delta_{22}/4 &= k^{-1} \sum_i [g^2(\underline{z}_i)] - 2u_2[k(k-1)]^{-1} \sum_{i \neq i'} \sum_{i' \neq i} h_2(\underline{z}_i, \underline{z}_{i'}) + u_2^2 \\ &= k^{-1} \sum_i [(k-1)^{-1} \sum_{i' \neq i} h_2(\underline{z}_i, \underline{z}_{i'})]^2 - u_2^2. \end{aligned}$$

Rewriting the first term we obtain

$$\begin{aligned} &[k(k-1)^2]^{-1} \left(\sum_{i \neq i'} \sum_{i' \neq i} h_2^2(\underline{z}_i, \underline{z}_{i'}) + 2 \sum_{i < i'} \sum_{i' < i} h_2(\underline{z}_i, \underline{z}_{i'}) h_2(\underline{z}_i, \underline{z}_{i'}) \right) \\ &= (k-1)^{-1} 2[k(k-1)]^{-1} \sum_{i < i'} \sum_{i' < i} h_2^2(\underline{z}_i, \underline{z}_{i'}) \\ (C.8) \quad &+ (k-2)(k-1)^{-1} 6[k(k-1)(k-2)]^{-1} \left(\sum_{i=1}^{i-1} \sum_{i'=i+1}^k h_2(\underline{z}_i, \underline{z}_{i'}) h_2(\underline{z}_i, \underline{z}_{i'}) \right. \\ &\quad \left. \sum_{i \neq i'} \right) \end{aligned}$$

$$+ \sum_{i < i'} \sum_{i'' < i} h_2(z_i, z_{i'}) h_2(z_i, z_{i''}).$$

The first term of (C.8) is equal to $(k-1)^{-1}$ times a U-statistic and therefore converges almost surely to zero. This follows from Theorem 2.1.3 since $E[h_2^2(z_1, z_2)] < \infty$ due to the assumed finite fourth moments, ϕ_4 and η_4 , of the ε_{ij} and α_i . The second term of (C.8) can be rewritten as

$$\begin{aligned} & (k-2)(k-1)^{-1} 6[k(k-1)(k-2)3]^{-1} \left(\sum_i \sum_{i'=1}^{i-1} \sum_{i''=i'+1}^{i-1} h_2(z_i, z_{i'}) h_2(z_i, z_{i''}) \right. \\ & + \sum_i \sum_{i'=1}^{i-1} \sum_{i''=i+1}^k h_2(z_i, z_{i'}) h_2(z_i, z_{i''}) \\ & \left. + \sum_{i < i'} \sum_{i'' < i} h_2(z_i, z_{i'}) h_2(z_i, z_{i''}) \right) \\ & = (k-2)(k-1)^{-1} \binom{k}{3}^{-1} \sum_{i < i' < i''} (1/3) [h_2(z_{i''}, z_i) h_2(z_{i''}, z_{i'}) \\ & + h_2(z_{i'}, z_i) h_2(z_{i'}, z_{i''}) + h_2(z_i, z_{i'}) h_2(z_i, z_{i''})], \end{aligned}$$

which is $(k-2)(k-1)^{-1}$ times a U-statistic and thus, again using Theorem 2.1.3, converges almost surely to $E[h_2(z_1, z_2)h_2(z_1, z_3)]$. Thus, (C.8) converges almost surely to $E[h_2(z_1, z_2)h_2(z_1, z_3)]$. The result is proven by using this fact and applying (C.2) with $c = 2$.

Using Results C.1, C.2, and C.3, and (C.2) and (C.3) with various values of c , we obtain a consistent estimate of σ_T^2 . We will denote the estimate by $\hat{\sigma}_T^2$ where

$$\hat{\sigma}_T^2 = \hat{\sigma}_{11} U_2^{-2} + \hat{\sigma}_{22} U_1^2 U_2^{-4} - 2\hat{\sigma}_{12} U_1 U_2^{-3}.$$

APPENDIX D
DERIVATION OF ENDPOINTS IN CHI-SQUARE PROCEDURE

Using the model and assumptions in Section 2.2 a confidence interval for ρ was derived using the χ^2_2 distribution (2.2.14). The formulas for the endpoints of this interval are derived in this appendix.

To find the slopes, d_1 and d_2 , of the two lines in Figure 2.2.1 we rewrite (2.2.13) as

$$\begin{aligned} \sigma_{22}X'^2 - 2\sigma_{22}U_1X' + \sigma_{22}U_1^2 + \sigma_{11}Y'^2 - 2\sigma_{11}U_2Y' + \sigma_{11}U_2^2 \\ - 2\sigma_{12}X'Y' + 2\sigma_{12}U_1Y' + 2\sigma_{12}U_2X' - 2\sigma_{12}U_1U_2 \\ - D\chi^2_{2,\zeta}k^{-1} = 0. \end{aligned}$$

Substituting $Y' = dX'$ and collecting coefficients yields

$$\begin{aligned} (D.1) \quad & X'^2(\sigma_{22} + \sigma_{11}d^2 - 2\sigma_{12}d) \\ & + X'(-2\sigma_{22}U_1 - 2\sigma_{11}U_2d + 2\sigma_{12}U_1d + 2\sigma_{12}U_2) \\ & + (\sigma_{22}U_1^2 + \sigma_{11}U_2^2 - 2\sigma_{12}U_1U_2 - D\chi^2_{2,\zeta}k^{-1}) = 0. \end{aligned}$$

The values of d for which $Y' = dX'$ are tangents to the ellipse depicted in Figure 2.2.1 are the values that yield only one solution of this quadratic equation in X' . If we write (D.1) as $a_1X'^2 + b_1X' + c_1 = 0$, the values of d we are seeking must satisfy $b_1^2 - 4a_1c_1 = 0$. Now,

$$\begin{aligned}
b_1^2 = & \ 4\sigma_{22}^2 U_1^2 + 4\sigma_{11}^2 U_2^2 d^2 + 4\sigma_{12}^2 U_1^2 d^2 + 4\sigma_{12}^2 U_2^2 \\
& + 8\sigma_{11}\sigma_{22}U_1U_2d - 8\sigma_{12}\sigma_{22}U_1^2d - 8\sigma_{12}\sigma_{22}U_1U_2 \\
& - 8\sigma_{11}\sigma_{12}U_1U_2d^2 - 8\sigma_{11}\sigma_{12}U_2^2d + 8\sigma_{12}^2U_1U_2d
\end{aligned}$$

and

$$\begin{aligned}
4a_1c_1 = & \ 4\sigma_{22}^2 U_1^2 + 4\sigma_{11}\sigma_{22}U_2^2 - 8\sigma_{12}\sigma_{22}U_1U_2 - 4\sigma_{22}Dx_{2\zeta}^2k^{-1} \\
& + 4\sigma_{11}\sigma_{22}U_1^2d^2 + 4\sigma_{11}^2U_2^2d^2 - 8\sigma_{11}\sigma_{12}U_1U_2d^2 \\
& - 4\sigma_{11}d^2Dx_{2\zeta}^2k^{-1} - 8\sigma_{12}\sigma_{22}U_1^2d - 8\sigma_{11}\sigma_{12}U_2^2d \\
& + 16\sigma_{12}^2U_1U_2d + 8\sigma_{12}dDx_{2\zeta}^2k^{-1},
\end{aligned}$$

hence,

$$\begin{aligned}
b_1^2 - 4a_1c_1 = & \ 4\sigma_{12}^2 U_1^2 d^2 + 4\sigma_{12}^2 U_2^2 + 8\sigma_{11}\sigma_{22}U_1U_2d \\
& - 8\sigma_{12}^2 U_1U_2d - 4\sigma_{11}\sigma_{22}U_2^2 + 4\sigma_{22}Dx_{2\zeta}^2k^{-1} \\
& - 4\sigma_{11}\sigma_{22}U_1^2d^2 + 4\sigma_{11}d^2Dx_{2\zeta}^2k^{-1} - 8\sigma_{12}dDx_{2\zeta}^2k^{-1} \\
(D.2) \quad = & \ d^2(4\sigma_{12}^2 U_1^2 - 4\sigma_{11}\sigma_{22}U_1^2 + 4\sigma_{11}Dx_{2\zeta}^2k^{-1}) \\
& + d(8\sigma_{11}\sigma_{22}U_1U_2 - 8\sigma_{12}^2 U_1U_2 - 8\sigma_{12}Dx_{2\zeta}^2k^{-1}) \\
& + (4\sigma_{12}^2 U_2^2 - 4\sigma_{11}\sigma_{22}U_2^2 + 4\sigma_{22}Dx_{2\zeta}^2k^{-1}).
\end{aligned}$$

Writing (D.2) as $a_2d^2 + b_2d + c_2$, we see the values of d that make $b_2^2 - 4a_1c_1$ equal to 0 are the roots of $a_2d^2 + b_2d + c_2 = 0$. These two roots are $(2a_2)^{-1}[-b_2 - (b_2^2 - 4a_2c_2)^{1/2}] \equiv r^-$ and $(2a_2)^{-1}[-b_2 + (b_2^2 - 4a_2c_2)^{1/2}] \equiv r^+$.

The values needed to find r^- and r^+ are

$$-b_2 = 8(-\sigma_{11}\sigma_{22}U_1U_2 + \sigma_{12}^2U_1U_2 + \sigma_{12}Dx_{2\zeta}^2k^{-1}),$$

$$\begin{aligned}
2a_2 &= 8(\sigma_{12}^2 U_1^2 - \sigma_{11}\sigma_{22}U_1^2 + \sigma_{11}Dx_{2\zeta}^2 k^{-1}), \\
b_2^2 &= 64(\sigma_{11}^2 \sigma_{22}^2 U_1^2 U_2^2 + \sigma_{12}^4 U_1^2 U_2^2 + \sigma_{12}^2 D^2 (x_{2\zeta}^2)^2 k^{-2} \\
&\quad - 2\sigma_{11}\sigma_{12}^2 \sigma_{22} U_1^2 U_2^2 - 2\sigma_{11}\sigma_{12}\sigma_{22} U_1 U_2 Dx_{2\zeta}^2 k^{-1} \\
&\quad + 2\sigma_{12}^3 U_1 U_2 Dx_{2\zeta}^2 k^{-1}), \\
4a_2 c_2 &= 64(\sigma_{12}^4 U_1^2 U_2^2 - \sigma_{11}\sigma_{12}^2 \sigma_{22} U_1^2 U_2^2 + \sigma_{12}^2 \sigma_{22} U_1^2 Dx_{2\zeta}^2 k^{-1} \\
&\quad - \sigma_{11}\sigma_{12}^2 \sigma_{22} U_1^2 U_2^2 + \sigma_{11}^2 \sigma_{22}^2 U_1^2 U_2^2 \\
&\quad - \sigma_{11}\sigma_{22}^2 U_1^2 Dx_{2\zeta}^2 k^{-1} + \sigma_{11}\sigma_{12}^2 U_2^2 Dx_{2\zeta}^2 k^{-1} \\
&\quad - \sigma_{11}^2 \sigma_{22}^2 U_2^2 Dx_{2\zeta}^2 k^{-1} + \sigma_{11}\sigma_{22} D^2 (x_{2\zeta}^2)^2 k^{-2}),
\end{aligned}$$

and

$$\begin{aligned}
b_2^2 - 4a_2 c_2 &= 64Dx_{2\zeta}^2 k^{-1} (\sigma_{12}^2 Dx_{2\zeta}^2 k^{-1} - 2\sigma_{11}\sigma_{12}\sigma_{22} U_1 U_2 \\
&\quad + 2\sigma_{12}^3 U_1 U_2 - \sigma_{12}^2 \sigma_{22} U_1^2 + \sigma_{11}\sigma_{22}^2 U_1^2 \\
&\quad - \sigma_{11}\sigma_{12}^2 U_2^2 + \sigma_{11}^2 \sigma_{22}^2 U_2^2 - \sigma_{11}\sigma_{22} D^2 x_{2\zeta}^2 k^{-1}).
\end{aligned}$$

Ideally, both r^- and r^+ will be greater than one since that would produce a confidence interval with endpoints between 0 and 1 (the range of possible values for ρ). If this occurs, we define the values of d as

$$d_1 = \min(r^-, r^+) \quad \text{and} \quad d_2 = \max(r^-, r^+).$$

In practice however, it is possible that one or both of r^- and r^+ are not greater than one. These situations are handled in the following manner. If the ellipse intersects the Y' axis, d_2 is set equal to ∞ . If the ellipse intersects the line $X' = Y'$, d_1 is set equal to 1. If both of these events occur, the confidence interval will have endpoints of 0 and 1. If only one occurs, the other value of d is set equal to the value of r^- or r^+ , whichever is greater than one.

It is also possible that $U_1 < U_2$ and the ellipse does not intersect the line $X' = Y'$. In this case both d_1 and d_2 are set equal to 1 and the endpoints of the interval are both 0.

Since σ_{11} , σ_{22} , and σ_{12} are unknown, we will replace them with the strongly consistent estimates derived in Appendix C. The values of d_1 and d_2 obtained using the estimates will still yield an asymptotic $100(1-\zeta)\%$ confidence interval for ρ of the form given in (2.2.14) using Slutsky's Theorem (Serfling 1980, Page 19).

APPENDIX E
C AND C^{*} TERMS

In this appendix it is shown that C_{1N} through C_{4N} ((3.2.9.a) to (3.2.9.d)) and C_{1N}^* through C_{4N}^* ((3.2.18.a) to (3.2.18.d)) are all $O_p(N^{-1/2})$. This is required to complete the proof of Theorem 3.2.1. As in the theorem, in this section we assume that the ϵ_{1j} and the α_i are independent observations from distributions that are symmetric about zero with distribution functions $F(x)$ and $G(x)$ and density functions $f(x)$ and $g(x)$ respectively. Also, δ_1 and δ_2 are scale parameters for the ϵ_{1j} and the α_i respectively with $\delta_1/\delta_2 = 1$ (implying $F(x) = G(x)$). It is further assumed that there exist finite constants, B_1 and B_2 , such that $F'(x) = f(x) < B_1$ for all x and $|F''(x)| = |f'(x)| < B_2$ for all x , that $f(0) > 0$, and that $\int x^2 f(x) dx < \infty$. Finally, we also recall the definitions of $H(x)$, $J_N[H_N(x)]$, $J[H(x)]$, $J'[H(x)]$, $F_n^*(x)$, $G_k^*(x)$, and $H_N^*(x)$ as given in (3.2.1), (3.2.2), (3.2.3), (3.2.4), (3.2.12), (3.2.13), and (3.2.14) respectively.

We begin by establishing some results that will be useful in the work that follows.

Proposition E.1. If $F(x)$ is the distribution function of the ϵ_{1j} and if T_N is a statistic such that $N^{1/2}T_N = O_p(1)$, then

$$\sup_x N^{1/2} |F(x+T_N) - F(x)| = O_p(1).$$

Proof: Using a Taylor series expansion we see

$$\begin{aligned}
 & \sup_x N^{1/2} |F(x+T_N) - F(x)| \\
 &= \sup_x N^{1/2} |T_N f(x+\tau_N)| \\
 &\quad \text{where } \tau_N \text{ is between 0 and } T_N \\
 &< B_1 N^{1/2} |T_N| \\
 &= o_p(1).
 \end{aligned}$$

Proposition E.2. If $F_n(x)$ is the empirical distribution function of the ϵ_{1j} , then

$$\sup_x N^{1/2} |F_n(x) - F(x)| = o_p(1).$$

Proof: This proposition follows from Theorem A in Serfling (1980, Page 59) which states that for every n , there exists a finite positive constant c (not depending on $F(x)$), such that

$$P(\sup_x |F_n(x) - F(x)| > d) < c \exp(-2nd^2) \quad \text{for } d > 0.$$

Proposition E.3. If $F_n^*(x)$ is as defined in (3.2.12), then

$$\sup_x N^{1/2} |F_n^*(x) - F(x)| = o_p(1).$$

Proof: By adding and subtracting appropriate terms and applying the triangle inequality we obtain

$$\begin{aligned}
 & \sup_x N^{1/2} |F_n^*(x) - F(x)| \\
 &< \sup_x N^{1/2} |F_n(x + \bar{\epsilon}_{1.}) - F(x + \bar{\epsilon}_{1.})| \\
 &\quad + \sup_x N^{1/2} |F(x + \bar{\epsilon}_{1.}) - F(x)| \\
 &= o_p(1) + o_p(1)
 \end{aligned}$$

$$= o_p(1),$$

using Propositions E.2 and E.1.

Let $G^n(x)$ and $g^n(x)$ denote the distribution and density functions respectively for the $\alpha_i + \bar{\varepsilon}_{i.}$, $i = 1, 2, \dots, k$. Also, let $G_k^n(x) = k^{-1} \sum_i I(\alpha_i + \bar{\varepsilon}_{i.} < x)$ be the empirical distribution function for the $\alpha_i + \bar{\varepsilon}_{i.}$.

Proposition E.4. If $g^n(x)$ is the density function for the $\alpha_i + \bar{\varepsilon}_{i.}$, then $g^n(x)$ is uniformly bounded by B_1 .

Proof: Recall that under the assumptions of Theorem 3.2.1, $g(x) = f(x) < B_1$ for all x . Let $F^n(x)$ be the distribution function for the $\bar{\varepsilon}_{i.}$. Then, since the α_i and $\bar{\varepsilon}_{i.}$ are independent, $g^n(x) = \int g(x-y)dF^n(y) < B_1/F^n(x) = B_1$.

Proposition E.5. If $g^n(x)$ is as described in Proposition E.4, then

$$\sup_x |g^n(x) - g(x)| = o(1).$$

Proof: First note that $g^n(x) = \int g(x-y)dF^n(y) = E[g(x - \bar{\varepsilon}_{1.})]$. Then,

$$\begin{aligned} & \sup_x |g^n(x) - g(x)| \\ &= \sup_x |E[g(x - \bar{\varepsilon}_{1.})] - g(x)| \\ &\leq E[\sup_x |g(x - \bar{\varepsilon}_{1.}) - g(x)|] \\ &= E[\sup_x |(-\bar{\varepsilon}_{1.})g'(x + \tau_N)|] \\ &\quad \text{where } \tau_N \text{ is between 0 and } -\bar{\varepsilon}_{1.} \\ &\leq B_2 E|\bar{\varepsilon}_{1.}| \\ &= o(1). \end{aligned}$$

Proposition E.6. If $G^n(x)$ is the distribution function of the $\alpha_1 + \varepsilon_{i,1}$, then

$$\sup_x |G^n(x) - G(x)| = o(N^{-1/2}).$$

Proof: Let $R_n = n^{1/2} \varepsilon_{1,1}$. and have distribution function $H^n(x)$. From the assumptions on the distribution of the $\varepsilon_{1,j}$ we know $R_n \xrightarrow{d} N(0, \sigma_R^2)$, where $\sigma_R^2 < \infty$. Look at $\alpha_1 + n^{-1/2} R_n$ which by definition has distribution function $G^n(x)$. Then

$$G^n(x) = P(\alpha_1 + n^{-1/2} R_n < x) = \int G(x - n^{-1/2} r) dH^n(r)$$

and thus

$$G^n(x) - G(x) = \int [G(x - n^{-1/2} r) - G(x)] dH^n(r).$$

Using a Taylor expansion,

$$G(x - n^{-1/2} r) = G(x) - n^{-1/2} r g(x) + (2n)^{-1} r^2 g'(x + \tau_n n^{-1/2} r),$$

where $0 < |\tau_n| < 1$. Therefore,

$$\begin{aligned} |G^n(x) - G(x)| &= |-n^{-1/2} g(x) \int r dH^n(r) + (2n)^{-1} \int r^2 g'(x + \tau_n n^{-1/2} r) dH^n(r)| \\ &\leq |-n^{-1/2} g(x) E(R_n)| + (2n)^{-1} B_2 \int r^2 dH^n(r), \end{aligned}$$

since $|g'(x)|$ is bounded by B_2 by assumption. Since $E(R_n) \xrightarrow{n \rightarrow \infty} 0$ (Chung 1968, Theorem 4.5.2) and $E(R_n^2)$ is uniformly bounded, we see that

$$\sup_x |G^n(x) - G(x)| \leq n^{-1/2} B_1 o(1) + (2n)^{-1} B_2 o(1) = o(N^{-1/2}).$$

Proposition E.7. If $G_k^*(x)$ is as defined in (3.2.13), then

$$\sup_x N^{1/2} |G_k^*(x) - G(x)| = o_p(1).$$

Proof: First, note that $G_k^*(x) = G_k^n(x+\bar{\alpha}+\bar{\varepsilon}_{\dots})$. Then, as in the proof of Proposition E.3, we add and subtract appropriate terms and apply the triangle inequality to obtain

$$\begin{aligned} & \sup_x N^{1/2} |G_k^*(x) - G(x)| \\ & \leq \sup_x N^{1/2} |G_k^n(x+\bar{\alpha}+\bar{\varepsilon}_{\dots}) - G^n(x+\bar{\alpha}+\bar{\varepsilon}_{\dots})| \\ & \quad + \sup_x N^{1/2} |G^n(x+\bar{\alpha}+\bar{\varepsilon}_{\dots}) - G(x+\bar{\alpha}+\bar{\varepsilon}_{\dots})| \\ & \quad + \sup_x N^{1/2} |G(x+\bar{\alpha}+\bar{\varepsilon}_{\dots}) - G(x)| \\ & = o_p(1) + o_p(1) + o_p(1) \\ & = o_p(1), \end{aligned}$$

using Proposition E.2 (since the constant c does not depend on the distribution function), Proposition E.6, and Proposition E.1.

Proposition E.8. If $H_N^*(x)$ is as defined in (3.2.14), then

$$\sup_x N^{1/2} |H_N^*(x) - H(x)| = o_p(1).$$

Proof: Using the definitions of $H_N^*(x)$ and $H(x)$ and again applying the triangle inequality, we obtain

$$\begin{aligned} & \sup_x N^{1/2} |H_N^*(x) - H(x)| \\ & \leq \sup_x \lambda_N N^{1/2} |F_n^*(x) - F(x)| \\ & \quad + \sup_x (1-\lambda_N) N^{1/2} |G_k^*(x) - G(x)| \\ & = o_p(1) + o_p(1) \\ & = o_p(1), \end{aligned}$$

using Propositions E.3 and E.7.

Recall now the pseudo-samples described in (3.1.1). Define \hat{X}_n to be the sample median of the X_j , $j = 1, 2, \dots, n$, and \hat{Y}_k to be the sample median of the Y_i , $i = 1, 2, \dots, k$. Also, let \hat{Z}_N be the sample median of the combined sample of the X_j and the Y_i . The following results establish some asymptotic properties of these sample medians.

Proposition E.9. If \hat{X}_n is the sample median of the X_j , then

$$N^{1/2} \hat{X}_n = O_p(1).$$

Proof: Due to the composition of the X_j we see that $\hat{X}_n = \hat{\epsilon}_1 - \bar{\epsilon}_{1..}$, where $\hat{\epsilon}_1 = \text{median of } (\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{1n})$. Since $N^{1/2} \hat{\epsilon}_1$ and $N^{1/2} \bar{\epsilon}_{1..}$ are both $O_p(1)$, it follows that $N^{1/2} \hat{X}_n = O_p(1)$.

Proposition E.10. If \hat{Y}_k is the sample median of the Y_i , then

$$N^{1/2} \hat{Y}_k = O_p(1).$$

Proof: Define $\hat{\alpha}$ to be the median of $(\alpha_1 + \bar{\epsilon}_{1..}, \alpha_2 + \bar{\epsilon}_{2..}, \dots, \alpha_k + \bar{\epsilon}_{k..})$. By the composition of the Y_i we see that $\hat{Y}_k = \hat{\alpha} - (\bar{\alpha} + \bar{\epsilon}_{..})$. Recall that the $\alpha_i + \bar{\epsilon}_{i..}$ are i.i.d. random variables, symmetrically distributed about zero, with distribution function $G^n(x)$. Using a theorem from Serfling (1980, Page 75) we know that for every $M > 0$,

$$P(|\hat{\alpha}| > M) \leq 2 \exp(-2k \Delta_M^2) \quad \text{for all } k,$$

where $\Delta_M = \min[G^n(M) - 1/2, 1/2 - G^n(M)]$. Since $G^n(x) = 1 - G^n(-x)$, it is seen that $\Delta_M = G^n(M) - 1/2 = G^n(M) - G^n(0)$. Thus,

$$\begin{aligned} P(N^{1/2} |\hat{\alpha}| < M) &= 1 - P(|\hat{\alpha}| > M/N^{1/2}) \\ &> 1 - 2 \exp(-2k [G^n(M/N^{1/2}) - G^n(0)]^2) \\ &= 1 - 2 \exp(-2k [(M^2/N) g^n(\tau_N)]), \end{aligned}$$

where τ_N is between 0 and $M/N^{1/2}$. Since $\tau_N \rightarrow 0$, we see from Proposition E.5 that $\lim_{N \rightarrow \infty} g^N(\tau_N) \rightarrow g(0)$. Thus, for large enough N , $g^N(\tau_N) > [g(0)/2]^2$. Therefore, for large enough N , we obtain

$$P(N^{1/2}|\hat{\alpha}| \leq M) > 1 - 2\exp[-(1/2)M^2g^2(0)\lambda_0],$$

where $\lambda_0 < k/N$ as described in Section 3.1. This inequality implies that for any $v > 0$, there exists $M > (-\ln(v/2)2[g^2(0)\lambda_0]^{-1})^{1/2}$, such that

$$P(N^{1/2}|\hat{\alpha}| \leq M) > 1 - v$$

for large enough N . This implies that $N^{1/2}\hat{\alpha} = o_p(1)$. Since $N^{1/2}(\bar{\alpha} + \bar{\epsilon}_n)$ is also $o_p(1)$, we see that $N^{1/2}\hat{Y}_k = o_p(1)$, completing the proof.

Proposition E.11. If \hat{Z}_N is the sample median of the combined sample of the X_j and the Y_i , then

$$N^{1/2}\hat{Z}_N = o_p(1).$$

Proof: Recalling the definitions of \hat{X}_n and \hat{Y}_k given previously, we see that

$$\min(\hat{X}_n, \hat{Y}_k) \leq \hat{Z}_N \leq \max(\hat{X}_n, \hat{Y}_k)$$

and therefore

$$0 < N^{1/2}|\hat{Z}_N| < N^{1/2}|\hat{X}_n| + N^{1/2}|\hat{Y}_k|.$$

The validity of the proposition follows from Propositions E.9 and E.10.

We now begin examining the C and C^* terms, beginning with C_{1N}^* . The following argument can also be used to show $C_{1N} = o_p(N^{-1/2})$ by replacing $F_n^*(x)$ with $F(x)$.

We recall that

$$C_{1N}^* = \lambda_N \int [F_n^*(x) - F(x)] J'[H(x)] d[F_n^*(x) - F(x)]$$

and show that

$$(E.1) \quad C_{1N}^* = (1/2) \lambda_N (\int_{\bar{R}} J'[H(x)] d[F_n^*(x) - F(x)]^2 + n^{-1} \int_{\bar{R}} J'[H(x)] d[F_n^*(x)]) .$$

If R is the set of points of increase of $F_n^*(x)$ and \bar{R} is the complement of R , then, as in Chernoff and Savage (1958, Page 987), the RHS of (E.1) can be written as

$$\begin{aligned} & (1/2) \lambda_N (\int_{\bar{R}} J'[H(x)] d[F_n^*(x) - F(x)]^2 + \int_{\bar{R}} J'[H(x)] d[F_n^*(x) - F(x)]^2 \\ & + n^{-2} \sum_j J'[H(X_j)]) \\ & = (1/2) \lambda_N (2 \int_{\bar{R}} J'[H(x)] [F_n^*(x) - F(x)] d[F_n^*(x) - F(x)] \\ & + \sum_j J'[H(X_j)] [(j/n - F(X_j))]^2 - [(j-1)/n - F(X_j)]^2 \\ & + n^{-2} \sum_j J'[H(X_j)]) \\ & = (1/2) \lambda_N (\int_{\bar{R}} J'[H(x)] [F_n^*(x) - F(x)] d[F_n^*(x) - F(x)] \\ & + \sum_j J'[H(X_j)] [(2/n)(j/n - F(X_j)) - n^{-2}] + n^{-2} \sum_j J'[H(X_j)]) \\ & = (1/2) \lambda_N (2 \int_{\bar{R}} J'[H(x)] [F_n^*(x) - F(x)] d[F_n^*(x) - F(x)] \\ & + 2 \sum_j J'[H(X_j)] [j/n - F(X_j)] [(j/n - F(X_j)) - ((j-1)/n - F(X_j))]) \\ & = \lambda_N (\int_{\bar{R}} J'[H(X_j)] [F_n^*(x) - F(x)] d[F_n^*(x) - F(x)] \\ & + \int_{\bar{R}} J'[H(x)] [F_n^*(x) - F(x)] d[F_n^*(x) - F(x)]) \\ & = \lambda_N \int [F_n^*(x) - F(x)] J'[H(x)] d[F_n^*(x) - F(x)] \\ & = C_{1N}^*, \end{aligned}$$

thus establishing (E.1). Thus,

$$\begin{aligned}
 c_{1N}^* &= \lambda_N/2 (\int_0^\infty d[F_n^*(x) - F(x)]^2 - \int_{-\infty}^0 d[F_n^*(x) - F(x)]^2 \\
 &\quad + n^{-2} \sum_j [H(X_j)]) \\
 &= \lambda_N/2 (-[F_n^*(0) - F(0)]^2 - [F_n^*(0) - F(0)]^2 + n^{-2} \sum_j [H(X_j)]) \\
 &= -\lambda_N [F_n^*(0) - F(0)]^2 + (\lambda_N/2) n^{-2} \sum_j [H(X_j)].
 \end{aligned}$$

Note that

$$\begin{aligned}
 |N^{1/2} n^{-2} \sum_j [H(X_j)]| &< N^{1/2} n^{-2} \sum_j |J'[H(X_j)]| \\
 &= N^{1/2}/n \\
 &= o(1)
 \end{aligned}$$

and that Proposition E.3 implies that $N^{1/2} [F_n^*(0) - F(0)]^2 = o_p(1)$.

Therefore,

$$\begin{aligned}
 N^{1/2} c_{1N}^* &= \lambda_N ((1/2) N^{1/2} n^{-2} \sum_j [H(X_j)] - N^{1/2} [F_n^*(0) - F(0)]^2) \\
 &= \lambda_N [o_p(1) + o_p(1)] \\
 &= o_p(1).
 \end{aligned}$$

Showing that c_{2N} and c_{2N}^* are $o_p(N^{-1/2})$ takes considerable work and for that reason we will delay looking at these terms until we have completed examination of the other terms.

Since $F_n^*(x)$ is the empirical distribution function of the X_j , we see that

$$c_{3N}^* = \int (J_N[H_N^*(x)] - J[H_N^*(x)]) dF_n^*(x)$$

$$= n^{-1} \sum_j (J_N^{H_N^*(X_j)} - J^{H_N^*(X_j)}).$$

Using the definitions of J_N and J we see that, with probability one,

$$\begin{aligned} n^{-1} \sum_j (|1/2 + (2N)^{-1} - H_N^*(X_j)| - |1/2 - H_N^*(X_j)|) &\leq c_{3N}^* < \\ n^{-1} \sum_j ((2N)^{-1} + |1/2 + (2N)^{-1} - H_N^*(X_j)| - |1/2 - H_N^*(X_j)|), \end{aligned}$$

which, after applying the triangle inequality, can be written as

$$\begin{aligned} n^{-1} \sum_j (|1/2 - H_N^*(X_j)| - |-(2N)^{-1}| - |1/2 - H_N^*(X_j)|) &\leq c_{3N}^* < \\ n^{-1} \sum_j ((2N)^{-1} + |1/2 + (2N)^{-1} - H_N^*(X_j)| - |1/2 + H_N^*(X_j)|). \end{aligned}$$

This inequality yields

$$-(2N)^{-1} \leq c_{3N}^* < N^{-1} \quad \text{with probability one,}$$

implying that $c_{3N}^* = o_p(N^{-1/2})$. The same proof can be used to show $c_{3N} = o_p(N^{-1/2})$ by replacing $F_n^*(x)$ and $H_N^*(x)$ with $F_n(x)$ and $H_N(x)$.

We now turn our attention to

$$c_{4N}^* = \int K_N^*(x) dF_n^*(x),$$

where $K_N^*(x)$ is as defined in Section 3.2. As before let \hat{Z}_N be the sample median of the combined sample of the X_j and the Y_i . Then,

$$\begin{aligned} c_{4N}^* &= \int [1 - 2H_N^*(x)] J'[H(x)] [I(0 < H_N^*(x) < 1/2) I(x > 0) \\ &\quad + I(1/2 < H_N^*(x) < 1) I(x < 0)] dF_n^*(x) \\ &= \int [1 - 2H_N^*(x)] I(0 < x < \hat{Z}_N) dF_n^*(x) + \int [2H_N^*(x) - 1] I(\hat{Z}_N < x < 0) dF_n^*(x) \\ &\leq 2 \int [1/2 - H_N^*(0)] I(0 < x < \hat{Z}_N) dF_n^*(x) + 2 \int [H_N^*(0) - 1/2] I(\hat{Z}_N < x < 0) dF_n^*(x) \\ &= 2 |H_N^*(0) - 1/2| |F_n^*(\hat{Z}_N) - F_n^*(0)|. \end{aligned}$$

This implies that

$$\begin{aligned}
 n^{1/2} C_{4N}^* &< 2n^{1/2} |H_N^*(0) - H(0)| [|F_n^*(\hat{z}_N) - F(\hat{z}_N)| \\
 &\quad + |F(\hat{z}_N) - F(0)| + |F(0) - F_n^*(0)|] \\
 &= o_p(1)[o_p(1) + o_p(1) + o_p(1)] \\
 &= o_p(1),
 \end{aligned}$$

using Propositions E.8, E.3, E.11, and the continuity of $F(x)$. The same argument can be used to show that $C_{4N} = o_p(N^{-1/2})$ by again replacing $F_n^*(x)$ and $H_N^*(x)$ with $F_n(x)$ and $H_N(x)$.

We now consider

$$(E.2) \quad C_{2N}^* = (1-\lambda_N) \int [G_k^*(x) - G(x)] J' [H(x)] d[F_n^*(x) - F(x)].$$

The following proof that $C_{2N}^* = o_p(N^{-1/2})$ involves steps similar to those used by Raghavachari (1965b) and Bhattacharyya (1977) in showing $C_{2N}^* = o_p(N^{-1/2})$ in slightly different situations. The proof that follows is more complex since the corresponding proofs that appear in the dissertations of Raghavachari (1965a) and Bhattacharyya (1973) are apparently incomplete. The proof that $C_{2N} = o_p(N^{-1/2})$ is a special case of the following argument, the main step being applying Lemma E.1 with $t_1 = t_2 = 0$.

Let $k' = k - 1$ and define $A_i = \alpha_{i+1} + \bar{\varepsilon}_{i+1},$ for $i = 1, 2, \dots, k'$. The A_i are independent and identically distributed as $A_1 = \alpha_2 + \bar{\varepsilon}_2,$ but the distribution of A_1 changes as N changes due to the presence of $\bar{\varepsilon}_2.$ Define $E_j = \varepsilon_{1j}$ for $j = 1, 2, \dots, n$ and note that our assumptions imply that the A_i and the E_j are independent. Also note that, as previously defined, $G^n(x)$ and $g^n(x)$ are the distribution and density functions for

the A_i and $F(x)$ and $f(x)$ are the distribution and density functions for the ϵ_{ij} .

Recall now that for any event A , $I(A)$ equals 1 if A occurs and 0 if A does not occur. The term C_{2N}^* can be written as

$$(E.3) \quad \begin{aligned} & (1-\lambda_N) \int [k^{-1} \sum_i I(\alpha_1 + \bar{\epsilon}_{1i} - \bar{\alpha} - \bar{\epsilon}_{..} < x) - G(x)] \times \\ & J'[H(x)] d[n^{-1} \sum_j I(\epsilon_{1j} - \bar{\epsilon}_{1.} < x) - F(x)]. \end{aligned}$$

We define a two-argument function as

$$(E.4) \quad \begin{aligned} C_{2N}^*(t_1, t_2) &= \int [k^{-1} \sum_{i=1}^{k'} I(A_i < x + t_1) - G(x)] \times \\ & J'[H(x)] d[n^{-1} \sum_j I(\epsilon_{1j} < x + t_2) - F(x)] \\ &= \int [k^{-1} \sum_{i=1}^{k'} I(A_i < x + t_1) - G(x)] J'[H(x)] d[F_n(x + t_2) - F(x)], \end{aligned}$$

where $F_n(x)$ is the empirical distribution function of the ϵ_{1j} as defined in Section 3.2.

Comparing (E.3) and (E.4) the relationship

$$(E.5) \quad \begin{aligned} C_{2N}^* &= (1-\lambda_N)(C_{2N}^*(\bar{\alpha} + \bar{\epsilon}_{..}, \bar{\epsilon}_{1.}) + \int k^{-1} I(\alpha_1 + \bar{\epsilon}_{1.} - \bar{\alpha} - \bar{\epsilon}_{..} < x) \times \\ & J'[H(x)] d[n^{-1} \sum_j I(\epsilon_{1j} - \bar{\epsilon}_{1.} < x) - F(x)]) \end{aligned}$$

is obtained. This relationship and the following lemma are used to show $C_{2N}^* = o_p(N^{-1/2})$.

Lemma E.1. For any fixed values, t_1 and t_2 , such that $|t_1|$ and $|t_2|$ are both bounded by some finite constant,

$$C_{2N}^*(N^{-1/2}t_1, N^{-1/2}t_2) = o_p(N^{-1/2}).$$

Proof: Let $\hat{G}_{k^-}(x) = k^{-1} \sum_{i=1}^{k^-} I(A_i < x)$ be the empirical distribution function of the A_i . To simplify notation let $t_{1N} = N^{-1/2} t_1$ and $t_{2N} = N^{-1/2} t_2$ and use (E.4) to write

$$\begin{aligned}
 C_{2N}^*(t_{1N}, t_{2N}) &= \int[(k-1)/k][\hat{G}_{k^-}(x+t_{1N}) - G(x)]J'[H(x)]d[F_n(x+t_{2N}) - F(x)] \\
 (E.6) \quad &= \int[\hat{G}_{k^-}(x+t_{1N}) - G^n(x)]J'[H(x)]d[F_n(x+t_{2N}) - F(x)] \\
 &\quad + \int[G^n(x) - G(x)]J'[H(x)]d[F_n(x+t_{2N}) - F(x)].
 \end{aligned}$$

Note that (E.4) and (E.6) are not exactly the same since \hat{G}_{k^-} in (E.6) differs from the analogous term in (E.4) by a factor of $k'/k = (k-1)/k$. This factor does not affect the asymptotic behavior of the term so it is ignored for the purpose of a clearer presentation.

Since $|J'[H(x)]| < 1$, Proposition E.6 implies that the second term on the RHS of (E.6) is $o_p(N^{-1/2})$. To complete the proof of the lemma we must show that the first term on the RHS of (E.6) is $o_p(N^{-1/2})$. We expand this term in a manner similar to that used by Bhattacharyya (1973) to obtain

$$\begin{aligned}
 &\int[\hat{G}_{k^-}(x+t_{1N}) - G^n(x)]J'[H(x)]d[F_n(x+t_{2N}) - F(x)] \\
 &= \int[\hat{G}_{k^-}(x+t_{1N}) - G^n(x+t_{1N})]J'[H(x)]d[F_n(x+t_{2N}) - F(x+t_{2N})] \\
 &\quad + \int[\hat{G}_{k^-}(x+t_{1N}) - G^n(x+t_{1N})]J'[H(x)]d[F(x+t_{2N}) - F(x)] \\
 &\quad + \int[G^n(x+t_{1N}) - G^n(x)]J'[H(x)]d[F_n(x+t_{2N}) - F(x+t_{2N})] \\
 &\quad + \int[G^n(x+t_{1N}) - G^n(x)]J'[H(x)]d[F(x+t_{2N}) - F(x)] \\
 &\equiv C_{21N} + C_{22N} + C_{23N} + C_{24N}.
 \end{aligned}$$

The following four propositions, which show that C_{21N} through C_{24N} are all $o_p(N^{-1/2})$, complete the proof of Lemma E.1.

Proposition E.12. Under the conditions of Theorem 3.2.1 and Lemma E.1, $C_{21N} = o_p(N^{-1/2})$.

Proof: Recall that

$$C_{21N} = \int [\widehat{G}_{k'}(x+t_{1N}) - G^n(x+t_{1N})] J'[H(x)] d[F_n(x+t_{2N}) - F(x+t_{2N})]$$

and that $E_j = \epsilon_{1j}$ for $j = 1, 2, \dots, n$. Look at

$$\begin{aligned} & E(C_{21N}^2 | E_1, E_2, \dots, E_n) \\ &= E\left(2 \int \int_{x < y} [\widehat{G}_{k'}(x+t_{1N}) - G^n(x+t_{1N})] [\widehat{G}_{k'}(y+t_{1N}) - G^n(y+t_{1N})] J'[H(x)] \times \right. \\ & \quad J'[H(y)] d[F_n(x+t_{2N}) - F(x+t_{2N})] d[F_n(y+t_{2N}) - F(y+t_{2N})] \\ & \quad + \int \int_{x=y} [\widehat{G}_{k'}(x+t_{1N}) - G^n(x+t_{1N})] [\widehat{G}_{k'}(y+t_{1N}) - G^n(y+t_{1N})] J'[H(x)] \times \\ & \quad J'[H(y)] d[F_n(x+t_{2N}) - F(x+t_{2N})] d[F_n(y+t_{2N}) - F(y+t_{2N})] \\ & \quad \left. | E_1, E_2, \dots, E_n \right). \end{aligned}$$

Since the A_i are mutually independent and independent of the E_j and since $E[I(A_i < x)] = G^n(x)$, for $x < y$ we obtain

$$\begin{aligned} & E([\widehat{G}_{k'}(x+t_{1N}) - G^n(x+t_{1N})] [\widehat{G}_{k'}(y+t_{1N}) - G^n(y+t_{1N})] | E_1, E_2, \dots, E_n) \\ &= E\left(\left[k'^{-1} \sum_{i=1}^{k'} I(A_i < x+t_{1N}) - G^n(x+t_{1N})\right] \left[k'^{-1} \sum_{i=1}^{k'} I(A_i < y+t_{1N}) - G^n(y+t_{1N})\right]\right) \\ &= E\left(k'^{-2} \left[\sum_{i=1}^{k'} I(A_i < x+t_{1N})\right] \left[\sum_{i=1}^{k'} I(A_i < y+t_{1N})\right]\right) - G^n(x+t_{1N}) G^n(y+t_{1N}) \\ &= (k'-1) k'^{-1} G^n(x+t_{1N}) G^n(y+t_{1N}) + k'^{-1} G^n(x+t_{1N}) - G^n(x+t_{1N}) G^n(y+t_{1N}) \\ &= k'^{-1} G^n(x+t_{1N}) [1 - G^n(y+t_{1N})]. \end{aligned}$$

For $x = y$ the expectation above is the variance of the proportion of successes in a binomial experiment with k' trials and is thus equal to $k'^{-1} G^n(x+t_{1N}) [1 - G^n(x+t_{1N})]$. Therefore, $E(C_{21N}^2 | E_1, E_2, \dots, E_n)$ is

equal to

$$\begin{aligned}
 & 2k^{-1} \int_{x < y} G^n(x+t_{1N})[1 - G^n(y+t_{1N})] J'[H(x)] J'[H(y)] \times \\
 & \quad [dF_n(x+t_{2N}) dF_n(y+t_{2N}) - dF_n(x+t_{2N}) dF(y+t_{2N}) \\
 & \quad - dF(x+t_{2N}) dF_n(y+t_{2N}) + dF(x+t_{2N}) dF(y+t_{2N})] \\
 & \quad + (nk')^{-1} \int_x G^n(x+t_{1N})[1 - G^n(x+t_{1N})] dF_n(x+t_{2N}) \\
 & \equiv C_{21Na} + C_{21Nb},
 \end{aligned}$$

since the integral over the region $[x=y]$ is zero with respect to any continuous measure. Thus, $E(C_{21N}^2) = E[E(C_{21N}^2 | E_1, E_2, \dots, E_n)] = E(C_{21Na}) + E(C_{21Nb})$.

Let $M^n(x,y) = G^n(x+t_{1N})[1 - G^n(y+t_{1N})] J'[H(x)] J'[H(y)]$ and note that $-1 < M^n(x,y) < 1$ for all x, y , and n . The expected value of C_{21Na} is

$$\begin{aligned}
 & E[2(n^2 k')^{-1} \sum_{jj'} M^n(E_j - t_{2N}, E_{j'} - t_{2N}) I(E_j < E_{j'})] \\
 & - 2k'^{-1} \int_y \sum_j M^n(E_j - t_{2N}, y) I(E_j - t_{2N} < y) dF(y+t_{2N}) \\
 & - 2k'^{-1} \int_x \sum_j M^n(x, E_j - t_{2N}) I(x < E_j - t_{2N}) dF(x+t_{2N}) \\
 & + 2k'^{-1} \int_{x < y} [M^n(x,y) dF(x+t_{2N}) dF(y+t_{2N})].
 \end{aligned}$$

Since the integrand in each term above is bounded, we can exchange the order of expectation and integration. Since the E_j are independent and identically distributed, the above expectation is equal to

$$\begin{aligned}
 & [2k'^{-1} - 2(nk')^{-1}] \int_{x < y} [M^n(x,y) dF(x+t_{2N}) dF(y+t_{2N})] \\
 & - 2k'^{-1} \int_{x < y} [M^n(x,y) dF(x+t_{2N}) dF(y+t_{2N})]
 \end{aligned}$$

$$\begin{aligned}
&= 2k^{-1} \int_{x < y} M^n(x, y) dF(x+t_{2N}) dF(y+t_{2N}) \\
&\quad + 2k^{-1} \int_{x < y} M^n(x, y) dF(x+t_{2N}) dF(y+t_{2N}) \\
&= -2(nk^{-1})^{-1} \int_{x < y} M^n(x, y) dF(x+t_{2N}) dF(y+t_{2N}) \\
&= o(N^{-2}).
\end{aligned}$$

The last equality above is justified because $|M^n(x, y)| < 1$ for all x, y , and n .

The expected value of C_{21Nb} is

$$\begin{aligned}
&E((n^2 k^{-1})^{-1} \sum_j G^n(E_j - t_{2N} + t_{1N}) [1 - G^n(E_j - t_{2N} + t_{1N})]) \\
&= (nk^{-1})^{-1} \int_x G^n(x + t_{1N}) [1 - G^n(x + t_{1N})] dF(x + t_{2N}) \\
&= o(N^{-2}),
\end{aligned}$$

since the integrand is bounded for all x and n .

Therefore, both C_{21Na} and C_{21Nb} have expectations that are $o(N^{-2})$ which implies that $E(C_{21N}^2) = o(N^{-1})$. The proposition is established by using the Markov inequality (Chow and Teicher 1978, Page 88).

Proposition E.13. Under the assumptions of Theorem 3.2.1 and Lemma E.1, $C_{22N} = o_p(N^{-1/2})$.

Proof: Recall that

$$C_{22N} = \int [\hat{G}_{k^{-1}}(x + t_{1N}) - G^n(x + t_{1N})] J'[\bar{H}(x)] d[F(x + t_{2N}) - F(x)].$$

Proceeding as in the proof of Proposition E.12 we can write C_{22N}^2 as

$$\begin{aligned}
&2 \int_{x < y} [\hat{G}_{k^{-1}}(x + t_{1N}) - G^n(x + t_{1N})] [\hat{G}_{k^{-1}}(y + t_{1N}) - G^n(y + t_{1N})] \times \\
&\quad J'[\bar{H}(x)] J'[\bar{H}(y)] [f(x + t_{2N}) f(y + t_{2N}) - f(x + t_{2N}) f(y) \\
&\quad - f(x) f(y + t_{2N}) + f(x) f(y)] dx dy
\end{aligned}$$

and $E(C_{22N}^2)$ as

$$\begin{aligned} & 2k^{-1} \int_{x < y} \int G^n(x+t_{1N}) [1 - G^n(y+t_{1N})] J'[H(x)] J'[H(y)] \times \\ & [f(x+t_{2N}) f(y+t_{2N}) - f(x+t_{2N}) f(y) - f(x) f(y+t_{2N}) + f(x) f(y)] dx dy \\ & \equiv 2k^{-1} (a_N - b_N - c_N + d_N). \end{aligned}$$

We look at the four integrals involved in $E(C_{22N}^2)$, making appropriate changes of variable in each one. In the first integral we set $u = x + t_{2N}$ and $v = y + t_{2N}$ and obtain

$$\begin{aligned} a_N = & \int_{uv} \int G^n(u-t_{2N}+t_{1N}) [1 - G^n(v-t_{2N}+t_{1N})] J'[H(u-t_{2N})] \times \\ & J'[H(v-t_{2N})] I(u < v) f(u) f(v) du dv. \end{aligned}$$

In the second integral we set $u = x + t_{2N}$ and $v = y$ and obtain

$$\begin{aligned} b_N = & \int_{uv} \int G^n(u-t_{2N}+t_{1N}) [1 - G^n(v+t_{1N})] J'[H(u-t_{2N})] \times \\ & J'[H(v)] I(u-t_{2N} < v) f(u) f(v) du dv. \end{aligned}$$

In the third integral we set $u = x$ and $v = y + t_{2N}$ and obtain

$$\begin{aligned} c_N = & \int_{uv} \int G^n(u+t_{1N}) [1 - G^n(v-t_{2N}+t_{1N})] J'[H(u)] \times \\ & J'[H(v-t_{2N})] I(u < v-t_{2N}) f(u) f(v) du dv. \end{aligned}$$

Finally, in the fourth integral we set $u = x$ and $v = y$ and obtain

$$\begin{aligned} d_N = & \int_{uv} \int G^n(u+t_{1N}) [1 - G^n(v+t_{1N})] J'[H(u)] \times \\ & J'[H(v)] I(u < v) f(u) f(v) du dv. \end{aligned}$$

Since $G^n(x)$ converges uniformly to $G(x)$ by Proposition E.6 and t_{1N} and t_{2N} both converge to zero by construction, a_N , b_N , c_N , and d_N all

converge to the same finite limit. Thus, $E(C_{22N}^2) = o(N^{-1})$ and again using the Markov inequality it follows that $C_{22N} = o_p(N^{-1/2})$, thus proving the proposition.

Proposition E.14. Under the assumptions of Theorem 3.2.1 and Lemma E.1, $C_{23N} = o_p(N^{-1/2})$.

Proof: Recall that

$$C_{23N} = \int [g^n(x+t_{1N}) - g^n(x)] J'[H(x)] d[F_n(x+t_{2N}) - F(x+t_{2N})].$$

Using a Taylor expansion we write C_{23N} as

$$t_{1N} \int g^n(x+\tau_N t_{1N}) J'[H(x)] d[F_n(x+t_{2N}) - F(x+t_{2N})],$$

where $|\tau_N| < 1$. The expected value of C_{23N}^2 is

$$\begin{aligned} & E(t_{1N}^2 \int g^n(x+\tau_N t_{1N}) g^n(y+\tau_N t_{1N}) J'[H(x)] J'[H(y)] \times \\ & \quad d[F_n(x+t_{2N}) F_n(y+t_{2N}) - F_n(x+t_{2N}) F(y+t_{2N}) \\ & \quad - F(x+t_{2N}) F_n(y+t_{2N}) + F(x+t_{2N}) F(y+t_{2N})]) \\ &= t_{1N}^2 E(n^{-2} \sum_{j,j'} g^n(E_j - t_{2N} + \tau_N t_{1N}) g^n(E_{j'} - t_{2N} + \tau_N t_{1N}) \times \\ & \quad J'[H(E_j - t_{2N})] J'[H(E_{j'} - t_{2N})]) \\ &= - \int_n^{-1} \sum_j g^n(E_j - t_{2N} + \tau_N t_{1N}) J'[H(E_j - t_{2N})] g^n(y + \tau_N t_{1N}) \times \\ & \quad J'[H(y)] dF(y+t_{2N}) \\ & - \int_n^{-1} \sum_j g^n(E_j - t_{2N} + \tau_N t_{1N}) J'[H(E_j - t_{2N})] g^n(x + \tau_N t_{1N}) \times \\ & \quad J'[H(x)] dF(x+t_{2N}) \\ & + [\int_x g^n(x + \tau_N t_{1N}) J'[H(x)] dF(x+t_{2N})]^2. \end{aligned}$$

Following the same reasoning as in the proof of Proposition E.12 we interchange the order of expectation and integration and, using the

independence of the E_j , the expectation is equal to

$$\begin{aligned} t_{1N}^2 & \left((n-1)n^{-1} \left[\int g^n(x + \tau_N t_{1N}) J' [H(x)] dF(x + t_{2N}) \right]^2 \right. \\ & + n^{-1} \left[\int g^n(x + \tau_N t_{1N}) \right]^2 dF(x + t_{2N}) \\ & - \left. \left[\int g^n(x + \tau_N t_{1N}) J' [H(x)] dF(x + t_{2N}) \right]^2 \right) \\ & = t_{1N}^{n-1} \left(\left[\int g^n(x + \tau_N t_{1N}) \right]^2 dF(x + t_{2N}) \right. \\ & \left. - \left[\int g^n(x + \tau_N t_{1N}) J' [H(x)] dF(x + t_{2N}) \right]^2 \right). \end{aligned}$$

From Proposition E.4 we know that $g^n(x)$ is uniformly bounded for all n . It therefore follows that the quantity within the large parentheses above is uniformly bounded for all n . Since $t_{1N} = N^{-1/2} t_1$, it follows that $E(c_{23N}^2) = O(N^{-2})$ and again using the Markov inequality we obtain $c_{23N} = o_p(N^{-1/2})$.

Proposition E.15. Under the assumptions of Theorem 3.2.1 and Lemma E.1, $c_{24N} = o(N^{-1/2})$.

Proof: Recall that

$$c_{24N} = \int [G^n(x + t_{1N}) - G^n(x)] J' [H(x)] d[F(x + t_{2N}) - F(x)].$$

Again using a Taylor expansion we write c_{24N} as

$$t_{1N} \int g^n(x + \tau_N t_{1N}) J' [H(x)] [f(x + t_{2N}) - f(x)] dx,$$

where $|\tau_N| < 1$. Writing this quantity in two integrals and letting $u = x + t_{2N}$ in the first we obtain

$$\begin{aligned} t_{1N} & \left(\int g^n(u - t_{2N} + \tau_N t_{1N}) J' [H(u - t_{2N})] f(u) du \right. \\ & \left. - \int g^n(x + \tau_N t_{1N}) J' [H(x)] f(x) dx \right) \\ & \equiv t_{1N} (a_{24N} - b_{24N}). \end{aligned}$$

By Proposition E.5, $g^n(x)$ converges to $g(x)$ uniformly in x and by Proposition E.4, and since $|J'[H(x)]| < 1$, the integrals are bounded. Thus, since $t_{2N} = N^{-1/2}t_2$, a_{24N} and b_{24N} both converge to the same finite limit by the Lebesgue Dominated Convergence Theorem (Chow and Teicher 1978, Page 99). Since $t_{1N} = N^{-1/2}t_1$, $c_{24N} = o(N^{-1/2})$ thus proving Proposition E.15 and completing the proof of Lemma E.1.

We now return to C_{2N}^* which, from (E.4) and (E.5) can be written as

$$(E.7) \quad \begin{aligned} & (1-\lambda_N) \int [k^{-1} \sum_{i=1}^k I(A_i - \bar{\alpha} - \bar{\varepsilon}_{..} < x) - G(x)] J'[H(x)] d[n^{-1} \sum_j (E_j - \bar{\varepsilon}_1) < x] - F(x) \\ & + (1-\lambda_N) k^{-1} \int I(\bar{\alpha} + \bar{\varepsilon}_1 - \bar{\alpha} - \bar{\varepsilon}_{..} < x) J'[H(x)] d[F_n^*(x) - F(x)], \end{aligned}$$

where $F_n^*(x)$ is the empirical distribution function of the X_j as defined in (3.2.12).

With a bounded integrand the second term in (E.7) is clearly $O_p(k^{-1})$ and hence $o_p(N^{-1/2})$. Thus, it remains only to show that the first term of (E.7), which after dropping the $1-\lambda_N$ we will refer to as $C_{2N}^*(\bar{\alpha} + \bar{\varepsilon}_{..}, \bar{\varepsilon}_1)$ using the definition in (E.4), is $o_p(N^{-1/2})$. The method of proof we use is patterned after a method used by Randles (1982) and Sukhatme (1958).

By assumptions of Theorem 3.2.1, $N^{1/2}(\bar{\alpha} + \bar{\varepsilon}_{..})$ and $N^{1/2}(\bar{\varepsilon}_1)$ are $O_p(1)$. Thus, for fixed $\Delta > 0$ there exists a bounded, two-dimensional sphere D , of radius M , centered at the origin, such that $P[N^{1/2}(\bar{\alpha} + \bar{\varepsilon}_{..}, \bar{\varepsilon}_1) \in D] > 1 - \Delta/2$ for every N .

To show $C_{2N}^*(\bar{\alpha} + \bar{\varepsilon}_{..}, \bar{\varepsilon}_1) = o_p(N^{-1/2})$ we will show that, for any fixed $\omega > 0$, $\lim_{N \rightarrow \infty} P[|N^{1/2}C_{2N}^*(\bar{\alpha} + \bar{\varepsilon}_{..}, \bar{\varepsilon}_1)| > \omega] = 0$. Let $(t_1, t_2) = \underline{t}$ be a point in D , implying $|t_1|$ and $|t_2|$ are both less than M . Then, again letting $t_{1N} = N^{-1/2}t_1$ and $t_{2N} = N^{-1/2}t_2$,

$$\begin{aligned}
& P[|N^{1/2}C_{2N}^*(\bar{\alpha} + \bar{\varepsilon}_{\dots}, \bar{\varepsilon}_{1\dots})| > \omega] \\
& \leq P[\sup_{\underline{t} \in D} |N^{1/2}C_{2N}^*(t_{1N}, t_{2N})| > \omega] + P[N^{1/2}(\bar{\alpha} + \bar{\varepsilon}_{\dots}, \bar{\varepsilon}_{1\dots}) \notin D] \\
& \leq P[\sup_{\underline{t} \in D} |N^{1/2}C_{2N}^*(t_{1N}, t_{2N})| > \omega] + \Delta/2.
\end{aligned}$$

Let D_u , for $1 \leq u \leq U$, be a finite set of spheres with centers $(s_{u1}, s_{u2}) \in D$ and radii $\|\underline{D}_u\| \leq \omega(1928B_1)^{-1}$ (where B_1 is the bound on $f(x)$ as described in assumption (ii) of Theorem 3.2.1) such that

$$D \subset \bigcup_{u=1}^U D_u.$$

To show $C_{2N}^* = o_p(N^{-1/2})$ we must show

$$(E.8) \quad \lim_{N \rightarrow \infty} P[\sup_{\underline{t} \in D} |N^{1/2}C_{2N}^*(t_{1N}, t_{2N})| > \omega] = 0.$$

Note that $\sup_{\underline{t} \in D} |N^{1/2}C_{2N}^*(t_{1N}, t_{2N})| > \omega$ implies that $\sup_{\underline{s} \in D_u} |N^{1/2}C_{2N}^*(N^{-1/2}s_1, N^{-1/2}s_2)| > \omega$ for some $1 \leq u \leq U$. Thus, the LHS of

(E.8) is less than or equal to

$$\lim_{N \rightarrow \infty} \sum_{u=1}^U P[\sup_{\underline{s} \in D_u} |N^{1/2}C_{2N}^*(N^{-1/2}s_1, N^{-1/2}s_2)| > \omega].$$

Therefore, (E.8) is true if

$$(E.9) \quad \lim_{N \rightarrow \infty} P[\sup_{\underline{s} \in D_u} |N^{1/2}C_{2N}^*(N^{-1/2}s_1, N^{-1/2}s_2)| > \omega] = 0$$

for every sphere D_u . Now, letting $(s_{1N}, s_{2N}) = (N^{-1/2}s_1, N^{-1/2}s_2)$ and $(s_{u1N}, s_{u2N}) = (N^{-1/2}s_{u1}, N^{-1/2}s_{u2})$,

$$\begin{aligned}
& P[\sup_{\underline{s} \in D_u} |N^{1/2}C_{2N}^*(s_{1N}, s_{2N})| > \omega] \\
& = P[\sup_{\underline{s} \in D_u} N^{1/2} |C_{2N}^*(s_{1N}, s_{2N}) - C_{2N}^*(s_{u1N}, s_{u2N}) + C_{2N}^*(s_{u1N}, s_{u2N})| > \omega]
\end{aligned}$$

$$(E.10) \quad \begin{aligned} & \leq P[\sup_{\substack{s \in D_u}} N^{1/2} |C_{2N}^*(s_{1N}, s_{2N}) - C_{2N}^*(s_{u1N}, s_{u2N})| > \omega/2] \\ & + P[N^{1/2} |C_{2N}^*(s_{u1N}, s_{u2N})| > \omega/2]. \end{aligned}$$

Since $(s_{u1}, s_{u2}) \in D$, from Lemma E.1 we know that

$C_{2N}^*(s_{u1N}, s_{u2N}) = o_p(N^{-1/2})$ which implies that, for large N , the second probability in (E.10) has limit zero. Therefore, if the limit of the first probability in (E.10) can be shown to be zero, it follows that (E.9) is true, and thus C_{2N}^* is $o_p(N^{-1/2})$. This is proved in the following lemma.

Lemma E.2. For any D_u , $1 < u < U$,

$$\sup_{\substack{s \in D_u}} N^{1/2} |C_{2N}^*(s_{1N}, s_{2N}) - C_{2N}^*(s_{u1N}, s_{u2N})| = o_p(1).$$

Proof: To begin, we will look at $C_{2N}^*(t_{1N}, t_{2N})$, where $(t_{1N}, t_{2N}) = (N^{-1/2}t_1, N^{-1/2}t_2)$ as before. As noted in the proof of Lemma E.1, without affecting the asymptotic behavior we can replace k with k' and write $C_{2N}^*(t_{1N}, t_{2N})$ as

$$(E.11) \quad \int [k'^{-1} \sum_{i=1}^{k'} I(A_i < x + t_{1N}) - G(x)] J'[H(x)] d[F_n(x + t_{2N}) - F(x)].$$

Recalling that under assumption (i) of Theorem 3.2.1 $\theta = 1$, implying $F(x) = G(x) = H(x)$, and separating the four parts of (E.11) we obtain

$$\begin{aligned} & (nk')^{-1} \sum_{i=1}^{k'} \sum_j I(A_i - E_j < t_{1N} - t_{2N}) J'[F(E_j - t_{2N})] \\ & - n^{-1} \sum_j F(E_j - t_{2N}) J'[F(E_j - t_{2N})] \\ & - k'^{-1} \sum_{i=1}^{k'} \int I(A_i < x + t_{1N}) J'[F(x)] dF(x) \end{aligned}$$

$$+ \int F(x) J'[F(x)] dF(x) \\ \equiv I_1 + I_2 + I_3 + I_4.$$

Each of I_2 , I_3 , and I_4 can be written as a double sum similar to I_1 . In I_2 we need only divide by k' and sum over $i = 1, 2, \dots, k'$. Upon making the change of variable $u = F(x)$ and recalling that $J'(u)$ equals -1 for $0 < u < 1/2$ and is equal to 1 for $1/2 < u < 1$ (3.2.25), each integral in I_3 is seen to be equal to

$$(E.12) \quad \int_{F(A_i - t_{1N})}^1 J'(u) du = \begin{cases} F(A_i - t_{1N}) & \text{if } A_i < t_{1N} \\ 1 - F(A_i - t_{1N}) & \text{if } A_i > t_{1N}. \end{cases}$$

Summing over $j = 1, 2, \dots, n$ and dividing by n produces a double sum for I_3 . Again using the change of variable $u = F(x)$, I_4 becomes

$$\int_0^1 u J'(u) du = - \int_0^{1/2} u du + \int_{1/2}^1 u du = 1/4$$

and by dividing by nk' and summing over $i = 1, 2, \dots, k'$ and $j = 1, 2, \dots, n$, we obtain a double sum for I_4 .

Therefore, $C_{2N}^*(t_{1N}, t_{2N})$ can be written as

$$(nk')^{-1} \sum_{i=1}^{k'} \sum_j h(E_j, A_i, t_{1N}, t_{2N}),$$

where $h(E_j, A_i, t_{1N}, t_{2N})$ is made up of four terms, one contributed by each of the double sums I_1 through I_4 . The exact form of $h(E_j, A_i, t_{1N}, t_{2N})$ depends on the relationships between the arguments. For example, suppose $A_i - E_j < t_{1N} - t_{2N}$, $E_j < t_{2N}$, and $A_i < t_{1N}$. The term contributed from I_1 would be -1 , I_2 would contribute $F(E_j - t_{2N})$, I_3 would contribute $-F(A_i - t_{1N})$, and I_4 would contribute $1/4$ (as it always does). Thus, in this case, $h(E_j, A_i, t_{1N}, t_{2N})$ would equal $-1 + F(E_j - t_{2N}) - F(A_i - t_{1N})$

+ 1/4. Looking at all possible relationships and eliminating those which are impossible (such as $A_i - E_j < t_{1N} - t_{2N}$, $E_j < t_{2N}$, and $A_i > t_{1N}$) we see that $h(E_j, A_i, t_{1N}, t_{2N})$ is equal to

$$(E.13) \quad \left\{ \begin{array}{l} -1 + F(E_j - t_{2N}) - F(A_i - t_{1N}) + 1/4 \\ \text{if } A_i - E_j < t_{1N} - t_{2N}, E_j < t_{2N}, A_i < t_{1N} \text{ (Area } W_1) \\ 1 - F(E_j - t_{2N}) - F(A_i - t_{1N}) + 1/4 \\ \text{if } E_j > t_{2N} \text{ and } A_i < t_{1N} \text{ (Area } W_2) \\ -F(E_j - t_{2N}) + F(A_i - t_{1N}) + 1/4 \\ \text{if } A_i - E_j < t_{1N} - t_{2N}, E_j > t_{2N}, A_i > t_{1N} \text{ (Area } W_3) \\ F(E_j - t_{2N}) - F(A_i - t_{1N}) + 1/4 \\ \text{if } A_i - E_j > t_{1N} - t_{2N}, E_j < t_{2N}, A_i < t_{1N} \text{ (Area } W_4) \\ -1 + F(E_j - t_{2N}) + F(A_i - t_{1N}) + 1/4 \\ \text{if } E_j < t_{2N} \text{ and } A_i > t_{1N} \text{ (Area } W_5) \\ -1 - F(E_j - t_{2N}) + F(A_i - t_{1N}) + 1/4 \\ \text{if } A_i - E_j > t_{1N} - t_{2N}, E_j > t_{2N}, A_i > t_{1N} \text{ (Area } W_6). \end{array} \right.$$

Since (s_1, s_2) and (s_{u1}, s_{u2}) are both points in the sphere D, all four coordinates are bounded in absolute value by M. Using the double sum form of $c_{2N}^*(t_{1N}, t_{2N})$ and returning to the term of interest we see that

$$\begin{aligned} & \sup_{\substack{s \in D_u}} N^{1/2} |c_{2N}^*(s_{1N}, s_{2N}) - c_{2N}^*(s_{u1N}, s_{u2N})| \\ & \leq N^{1/2} (nk')^{-1} \sum_{i=1}^{k'} \sum_{\substack{j \in D_u}} |h(E_j, A_i, s_{1N}, s_{2N}) - h(E_j, A_i, s_{u1N}, s_{u2N})| \\ & = N^{1/2} (nk')^{-1} \sum_{i=1}^{k'} \sum_{\substack{j \in D_u}} [\sup_{\substack{s \in D_u}} |h(E_j, A_i, s_{1N}, s_{2N}) - h(E_j, A_i, s_{u1N}, s_{u2N})| \\ & \quad - E[\sup_{\substack{s \in D_u}} |h(E_j, A_i, s_{1N}, s_{2N}) - h(E_j, A_i, s_{u1N}, s_{u2N})|]] \end{aligned}$$

$$+ N^{1/2} E[\sup_{\underline{s} \in D_u} |h(E_1, A_1, s_{1N}, s_{2N}) - h(E_1, A_1, s_{u1N}, s_{u2N})|]$$

$$\equiv D_{1N} + D_{2N}$$

and therefore the lemma is established by showing that

$$(E.14) \quad D_{1N} + D_{2N} = o_p(1).$$

We look first at

$$D_{2N} = N^{1/2} E[\sup_{\underline{s} \in D_u} |h(E_1, A_1, s_{1N}, s_{2N}) - h(E_1, A_1, s_{1N}, s_{2N})|].$$

The value of $h(E_1, A_1, s_{1N}, s_{2N}) - h(E_1, A_1, s_{u1N}, s_{u2N})$ will depend on which of the 36 possible regions the points (s_{1N}, s_{2N}) and (s_{u1N}, s_{u2N}) fall into. The 36 regions are combinations of the areas from (E.13) which will be denoted by $W_d(s_{1N}, s_{2N}) \cap W_m(s_{u1N}, s_{u2N})$ for $d = 1, 2, \dots, 6$ and $m = 1, 2, \dots, 6$.

For the six cases where $d = m$ the value of $|h(E_1, A_1, s_{1N}, s_{2N}) - h(E_1, A_1, s_{u1N}, s_{u2N})|$ is bounded by $2B_1 ||D_u|| N^{-1/2}$, as can be seen by considering the case $d = m = 1$. In this case, using (E.13),

$$\begin{aligned} & |h(E_1, A_1, s_{1N}, s_{2N}) - h(E_1, A_1, s_{u1N}, s_{u2N})| \\ &= |F(E_1 - s_{2N}) - F(E_1 - s_{u2N}) + F(A_1 - s_{u1N}) - F(A_1 - s_{1N})| \\ &\leq |F(E_1 - s_{2N}) - F(E_1 - s_{u2N})| + |F(A_1 - s_{u1N}) - F(A_1 - s_{1N})| \\ &= |(s_{u2N} - s_{2N})f[E_1 - s_{u2N} + \tau_N(s_{u2N} - s_{2N})]| \\ &\quad + |(s_{1N} - s_{u1N})f[A_1 - s_{1N} + \tau'_N(s_{1N} - s_{u1N})]|, \end{aligned}$$

where $|\tau_N|$ and $|\tau'_N|$ are both bounded by 1. Recalling that $(s_{1N}, s_{2N}) = (N^{-1/2}s_1, N^{-1/2}s_2)$ and $(s_{u1N}, s_{u2N}) = (N^{-1/2}s_{u1}, N^{-1/2}s_{u2})$, the above quantity is seen to be less than or equal to

$$\begin{aligned} & B_1 N^{-1/2} (|s_{u2} - s_2| + |s_1 - s_{u1}|) \\ & \leq 2B_1 ||D_u|| N^{-1/2}. \end{aligned}$$

In the thirty regions where $d \neq m$ it is not difficult to see that $|h(E_1, A_1, s_{1N}, s_{2N}) - h(E_1, A_1, s_{u1N}, s_{u2N})|$ is bounded by 8. The conditions on E_1 , A_1 , and $A_1 - E_1$ in each of these thirty regions are such that the probability of (s_{1N}, s_{2N}) and (s_{u1N}, s_{u2N}) being in any one of these is bounded by $2B_1 ||D_u|| N^{-1/2}$.

To see this in one case (the others are similar) let $H^n(x)$ be the distribution function of $A_1 - E_1$ and consider the region $A_1(s_{1N}, s_{2N}) \cap A_4(s_{u1N}, s_{u2N})$. The probability of this region is

$$\begin{aligned} & P(s_{u1N} - s_{u2N} < A_1 - E_1 \leq s_{1N} - s_{2N}, E_1 < s_{2N} \text{ and } s_{u2N}, A_1 < s_{1N} \text{ and } s_{u1N}) \\ & \leq P(s_{u1N} - s_{u2N} < A_1 - E_1 \leq s_{1N} - s_{2N}) \\ & = H^n(s_{1N} - s_{2N}) - H^n(s_{u1N} - s_{u2N}) \quad \text{or} \quad 0 \\ & \leq [(s_{1N} - s_{2N}) - (s_{u1N} - s_{u2N})] h^n[(s_{u1N} - s_{u2N}) + \tau_N(s_{1N} - s_{2N})], \end{aligned}$$

where $|\tau_N| \leq 1$ and $h^n(x) = H^{n-1}(x)$. In a method similar to that used in Proposition E.4 we can show that $h^n(x)$ is bounded by B_1 and therefore this probability is bounded by

$$\begin{aligned} & N^{-1/2} B_1 (|s_1 - s_2| + |s_{u1} - s_{u2}|) \\ & \leq 2B_1 ||D_u|| N^{-1/2}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \sup_{\underline{s} \in D_u} |h(E_1, A_1, s_{1N}, s_{2N}) - h(E_1, A_1, s_{u1N}, s_{u2N})| \\ (E.15) \quad & \leq \begin{cases} 8 & \text{for } E_1, A_1 \in \bigcup_{d \neq m}^6 [W_d(s_{1N}, s_{2N}) \cap W_m(s_{u1N}, s_{u2N})] \\ 2B_1 ||D_u|| N^{-1/2} & \text{for } E_1, A_1 \in \bigcup_{d=1}^6 [W_d(s_{1N}, s_{2N}) \cap W_d(s_{u1N}, s_{u2N})]. \end{cases} \end{aligned}$$

Thus,

$$\begin{aligned}
 D_{2N} &= N^{1/2} E[\sup_{\underline{s} \in D_u} |h(E_1, A_1, s_{1N}, s_{2N}) - h(E_1, A_1, s_{u1N}, s_{u2N})|] \\
 &\leq N^{1/2} [8(60B_1 ||D_u|| N^{-1/2}) + 2B_1 ||D_u|| N^{-1/2}] \\
 &= 482B_1 ||D_u|| \\
 &\leq 482B_1 \omega(1928B_1)^{-1} \\
 &= \omega/4,
 \end{aligned}$$

since $||D_u|| \leq \omega(1928B_1)^{-1}$ by construction. Thus $D_{2N} = o_p(1)$.

To show (E.14) is true it only remains to prove that $D_{1N} = o_p(1)$. From (E.14) we see that $E(D_{1N}) = 0$. Therefore, if we show that $E(D_{1N}^2) = \text{Var}(D_{1N}) = o(1)$, then we have shown $D_{1N} = o_p(1)$.

Letting

$$\begin{aligned}
 \hat{h}(E_j, A_1, s_N) &= \sup_{\underline{s} \in D_u} |h(E_j, A_1, s_{1N}, s_{2N}) - h(E_j, A_1, s_{u1N}, s_{u2N})| \\
 &- E[\sup_{\underline{s} \in D_u} |h(E_j, A_1, s_{1N}, s_{2N}) - h(E_j, A_1, s_{u1N}, s_{u2N})|],
 \end{aligned}$$

by combining like terms and remembering that the E_j are independent and identically distributed and are independent of the independent and identically distributed A_1 , we see that

$$\begin{aligned}
 E(D_{1N}^2) &= N(nk')^{-2} (nk') E[\hat{h}^2(E_1, A_1, s_N)] \\
 &\quad + n(n-1)k'E[\hat{h}(E_1, A_1, s_N)\hat{h}(E_2, A_1, s_N)] \\
 &\quad + nk'(k'-1)E[\hat{h}(E_1, A_1, s_N)\hat{h}(E_1, A_2, s_N)] \\
 &\quad + n(n-1)k'(k'-1)E[\hat{h}(E_1, A_1, s_N)\hat{h}(E_2, A_2, s_N)] \\
 &= N(nk')^{-1} (\text{Var}[\sup_{\underline{s} \in D_u} |h(E_1, A_1, s_{1N}, s_{2N}) - h(E_1, A_1, s_{u1N}, s_{u2N})|]) \\
 &\quad + (n-1)\text{Cov}[\sup_{\underline{s} \in D_u} |h(E_1, A_1, s_{1N}, s_{2N}) - h(E_1, A_1, s_{u1N}, s_{u2N})|, \\
 &\quad \quad \sup_{\underline{s} \in D_u} |h(E_2, A_1, s_{1N}, s_{2N}) - h(E_2, A_1, s_{u1N}, s_{u2N})|]
 \end{aligned}$$

$$+ (k'-1) \text{Cov}[\sup_{\underline{s} \in D_u} |h(E_1, A_1, s_{1N}, s_{2N}) - h(E_1, A_1, s_{u1N}, s_{u2N})|, \\ \sup_{\underline{s} \in D_u} |h(E_1, A_2, s_{1N}, s_{2N}) - h(E_1, A_2, s_{u1N}, s_{u2N})|],$$

since $E[h(E_j, A_1, s_N)] = 0$.

Using the fact that if $\text{Var}(X) = \text{Var}(Y)$, then $|\text{Cov}(X, Y)| \leq [\text{Var}(X)\text{Var}(Y)]^{1/2} = \text{Var}(X)$, we see that

$$\begin{aligned} E(D_{1N}^2) &\leq N(nk')^{-1} ((n+k'-1)\text{Var}[\sup_{\underline{s} \in D_u} |h(E_1, A_1, s_{1N}, s_{2N}) \\ &\quad - h(E_1, A_1, s_{u1N}, s_{u2N})|]) \\ &\leq N(n+k'-1)(nk')^{-1} E([\sup_{\underline{s} \in D_u} |h(E_1, A_1, s_{1N}, s_{2N}) - h(E_1, A_1, s_{u1N}, s_{u2N})|]^2) \\ &\leq N(n+k'-1)(nk')^{-1} [64(60B_1 ||D_u|| N^{-1/2}) + 4B_1^2 ||D_u||^2 N^{-1}] \\ &\xrightarrow[N \rightarrow \infty]{} 0, \end{aligned}$$

using (E.15) and the fact that $N(n+k'-1)(nk')^{-1} = O(1)$ due to the assumptions on the growth rate of n and k' . Therefore, we have shown $D_{1N} = o_p(1)$ which completes the proof of (E.14), and hence Lemma E.2. This also completes the proof that $C_{2N}^* = o_p(N^{-1/2})$.

REFERENCES

Anderson, K. (1981). Convergence of Moments of Standardized Quantiles. Unpublished Manuscript.

Ansari, A. R. and Bradley, R. A. (1960). Rank Sum Tests for Dispersion. Annals of Mathematical Statistics 31, 1174-1189.

Arvesen, J. N. (1969). Jackknifing U-statistics. Annals of Mathematical Statistics 40, 2076-2100.

Arvesen, J. N. and Layard, M. W. J. (1975). Asymptotically Robust Tests in Unbalanced Variance Component Models. Annals of Statistics 3, 1122-1134.

Arvesen, J. N. and Schmitz, T. H. (1970). Robust Procedures for Variance Component Problems Using the Jackknife. Biometrics 26, 677-686.

Bauer, D. F. (1972). Constructing Confidence Sets Using Rank Statistics. Journal of the American Statistical Association 67, 687-690.

Bhattacharyya, H. T. (1973). Nonparametric Estimates of Scale. Ph.D. Dissertation, University of North Carolina at Chapel Hill.

Bhattacharyya, H. T. (1977). Nonparametric Estimation of Ratio of Scale Parameters. Journal of the American Statistical Association 72, 459-463.

Bishop, Y. M. M., Fienberg, S. E., and Holland, P. W. (1975). Discrete Multivariate Analysis: Theory and Practice. The MIT Press, Cambridge, Massachusetts.

Chernoff, H. and Savage, I. R. (1958). Asymptotic Normality and Efficiency of Certain Nonparametric Test Statistics. Annals of Mathematical Statistics 29, 972-994.

Chow, Y. S. and Teicher, H. (1978). Probability Theory. Springer-Verlag, New York.

Chung, K. L. (1968). A Course in Probability Theory. Harcourt, Brace, and World, Inc., New York.

Govindarajulu, Z. (1975). Locally Most Powerful Rank Order Tests for the One-Way Random Effects Model. Studia Scientiarum Mathematicarum Hungarica 10, 47-60.

Govindarajulu, Z. and Deshpande', J. V. (1972). Random Effects Model: Nonparametric Case. Annals of the Institute of Statistical Mathematics 24, 165-170.

Hodges, J. L. and Lehmann, E. L. (1963). Estimates of Location Based on Rank Tests. Annals of Mathematical Statistics 34, 598-611.

Hoeffding, W. (1948). A Class of Statistics with Asymptotically Normal Distribution. Annals of Mathematical Statistics 19, 293-325.

Hoeffding, W. (1961). The Strong Law of Large Numbers for U-statistics. Institute of Statistics Mimeo Series No. 302, University of North Carolina.

Lehmann, E. L. (1963). Nonparametric Confidence Interval for a Shift Parameter. Annals of Mathematical Statistics 34, 1507-1512.

Raghavachari, M. (1965a). The Two-Sample Scale Problem when Locations are Unknown. Ph.D. Dissertation, University of California, Berkeley.

Raghavachari, M. (1965b). The Two-Sample Scale Problem when Locations are Unknown. Annals of Mathematical Statistics 36, 1236-1242.

Randles, R. H. (1982). On the Asymptotic Normality of Statistics with Estimated Parameters. Annals of Statistics 10, 462-474.

Scheffe', H. (1959). The Analysis of Variance. Wiley, New York.

Sen, P. K. (1963). On the Estimation of Relative Potency In Dillution (-Direct) Assays by Distribution-Free Methods. Biometrics 19, 532-552.

Sen, P. K. (1966). On a Distribution-Free Method of Estimating Asymptotic Efficiency of a Class of Nonparametric Tests. Annals of Mathematical Statistics 37, 1759-1770.

Serfling, R. J. (1980). Approximation Theorems of Mathematical Statistics. Wiley, New York.

Shoemaker, L. H. (1981). Robust Estimation and Testing in the One Way Random Effects Model. Unpublished Manuscript.

Snedecor, G. W. and Cochran, W. G. (1967). Statistical Methods, 6th edition. The Iowa State University Press, Ames, Iowa.

Sukhatme, B. V. (1958). Testing the Hypotheses that Two Populations Differ Only in Location. Annals of Mathematical Statistics 29, 60-78.

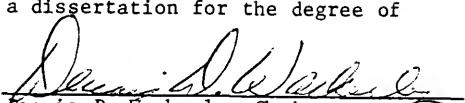
Tukey, J. W. (1958). Bias and Confidence in Not-Quite Large Samples (abstract). Annals of Mathematical Statistics 29, 614.

BIOGRAPHICAL SKETCH

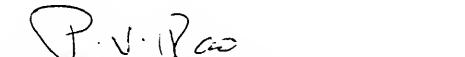
David John Groggel was born on November 1, 1956, in Grand Rapids, Michigan. He attended Ottawa Hills High School and Calvin College, receiving a Bachelor of Science degree in mathematics from Calvin in May, 1978.

In September, 1978, David entered graduate school in the Department of Statistics at the University of Florida. In December, 1979, he married Kathy Rubin of Ocala, Florida. He received a Master of Statistics degree from the University of Florida in June, 1980, and has since been pursuing a doctoral degree in statistics. As a graduate student David has taught numerous statistics courses at various levels.

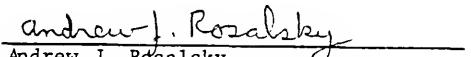
I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.


Dennis D. Wackerly, Chairman
Associate Professor of Statistics

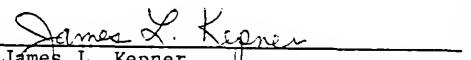
I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.


P. V. Rao
Professor of Statistics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.


Andrew J. Rosalsky
Assistant Professor of Statistics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.


James L. Kepner
Assistant Professor of Statistics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.


Thom J. Hodgson
Professor of Industrial and
Systems Engineering

This dissertation was submitted to the Graduate Faculty of the Department of Statistics in the College of Liberal Arts and Sciences and to the Graduate School, and was accepted for partial fulfillment of the requirements of the degree of Doctor of Philosophy.

Dean for Graduate Studies
and Research

UNIVERSITY OF FLORIDA



3 1262 08666 913 1